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A FUNDAMENTAL PROPERTY OF MARKOV PROCESSES WITH AN APPLICATION TO EQUIVALENCE UNDER TIME CHANGES

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This paper is dedicated to the memory of Shlomo Horowitz

ABSTRACT

It is shown that Markov processes traverse their trajectories in just one way, and applications are given to the Blumenthal, Getoor and McKean theorem.

§1. Introduction

There are two main results we wish to state which are proved in part in the present paper and in full in forthcoming papers. The first is that whenever a particle which is undergoing motion governed by a strong Markov process with stationary transition probabilities and without holding points traces out a fixed trajectory segment, it must trace it out in a fixed length of time depending only on the transition probabilities and on the trajectory segment. This theorem has as an immediate corollary the fact that a continuous Markov process on the real line which always moves in the same direction must do so deterministically. The second main result, an application of the first, is that if two processes have the same hitting probabilities and the second is a strong Markov process with stationary transition probabilities, then the first can be transformed into the second by a non-anticipating change of the time scale, without having to enlarge the σ-fields. This is a sharpening (in the case that the second process is Markov) of the result given in [2] for two arbitrary processes with no holding points.

As is customary in probability theory, *path* denotes a function of time having values in the state space and *trajectory* denotes the ordered set of points traversed by the path, so that, for example, if ω is fixed, $X_t(\omega)$ and $X_{t^2}(\omega)$ are different paths traversing the same trajectory.

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To carry out the proof of the first result, let $\tau_1 \leqq \tau_2$ be two stopping times, with τ_2 defined by the post-τ_1 behavior of the process. First, the paths of the Markov process are grouped into equivalence classes consisting of those paths which traverse the same trajectory segment as t ranges throughout the interval $[\tau_1, \tau_2)$. The probability measure of the process is then disintegrated with respect to these equivalence classes. Next, it is shown that the time of traversal of the segment is, for almost all trajectory segments, a constant depending on the trajectory segment and the transition probabilities of the process. This dependence is such that the traversal time of trajectory segments is consistent in the sense that if a segment is the union of other segments, then its traversal time is the sum of the traversal times of its components. That the traversal time of a segment is constant is proved by first showing that it can be written as the sum of an arbitrarily large number of random variables, which, relative to the conditional measure on the trajectory segment, are not only independent but on $\{\tau_2 < \infty\}$ are subject to the normality convergence criterion for triangular arrays, and is therefore normal. Since the traversal time must be non-negative, its normal distribution must degenerate at a constant. The idea of the proof that the traversal time is the sum of independent random variables is contained in the following observations. Let σ_1 be the first exit time from a ball of radius 1 centered at the location of the process at time 0, and let σ_2 be the first exit time after σ_1 from a ball of radius 1 centered at the location of the process at time σ_1. The paths are grouped into equivalence classes indexed by the three coordinates $\xi = (X_1, X_2, X_3)$, where X_1 is the location of the path at time 0, X_2 the location of the path at time σ_1 and X_3 the location of the path at time σ_2, and then the measure P on path space is disintegrated with respect to the equivalence classes. The Markov property as usually stated would give the conditional independence of σ_1 and $\sigma_2 - \sigma_1$ given X_2, but what is needed to give the independence of σ_1 and $\sigma_2 - \sigma_1$ with respect to P_ξ for almost all ξ is the conditional independence of σ_1 and $\sigma_2 - \sigma_1$ given $\xi = (X_1, X_2, X_3)$. This follows from the first part on applying the lemma that if $\mathscr{A}$ and $\mathscr{B}$ are conditionally independent given $\mathscr{C}$, and if $\mathscr{A}_0 \subset \mathscr{A}$, $\mathscr{B}_0 \subset \mathscr{B}$, $\mathscr{C}_0 \subset \mathscr{C}$ then $\mathscr{A}$ is conditionally independent of $\mathscr{B} \vee \mathscr{C}_0$ given $A_0 \vee \mathscr{B}_0 \vee \mathscr{C}$.

The second main result can be given in stronger form when both processes are strong Markov with stationary transition probability functions. It can be shown in that case that the time change satisfies a functional equation from which it follows that the transformed process is Markov with respect to the transformed σ-fields, a minor strengthening of the Blumenthal, Getoor and McKean theorem.

If the condition that there are no holding points is removed, then it can be shown that the first result still holds when properly modified. The modification takes into account the fact that for each holding point of the Markov process there is a parameter, depending only on the point, such that when the process reaches the point, it holds for an exponential holding time with that parameter. The process thus has more than one sample path in each trajectory segment having holding points. When restricted to a trajectory, the process will move deterministically when it moves, and will hold exponentially at holding points. The first result thus modified is what is needed to yield the Blumenthal, Getoor and McKean theorem directly.

The present paper contains a proof of the partial result that if paths tracing out the same full trajectory are grouped into equivalence classes, and if the measure is disintegrated with respect to these equivalence classes, then the disintegrated measure is concentrated on a single path per trajectory. This holds if both processes are strong Markov processes with stationary transition probabilities, and there are holding points. This result is then applied to yield the second main result in the case that there are no holding points as well as the Blumenthal, Getoor and McKean theorem in that case.

§2. Notation and preliminaries

Let (S, d) be a locally compact, separable metric space (S is then σ-compact, that is, S is a countable union of compact subsets). We denote by Σ the σ-field generated by the open subsets of S. Let D be the set of all S-valued functions on $[0,\infty)$ which are everywhere continuous from the right with limits (in S) from the left. There is a metric on D relative to which D is a complete separable metric space. Let $\mathcal{D}$ be the σ-field over D generated by the corresponding open sets. It turns out that $\mathcal{D}$ is also the σ-field generated by the maps $f \to f(t)$ as t varies over $[0,\infty)$. For each $s \geqq 0$, let $\mathcal{D}_s$ be the sub-σ-field of $\mathcal{D}$ generated by the maps $f \to f(t)$, $t \in [0, s]$ and $f \in D$. Let $\mathcal{D}_{s^+} = \bigcap_{t>s} \mathcal{D}_t$. We adjoin to S a non-member Δ as an isolated point, and set $f(\infty) = \Delta$ for each $f \in D$. A map ρ of D into the extended reals is called a *path-defined stopping time* if $\{f : \rho(f) \leqq s\} \in \mathcal{D}_s$ for each $s \geqq 0$. We write $f(\rho)$ for $f(\rho(f))$. The σ-field generated by the maps $f \to f(\rho + t)$ as t ranges over $[0,\infty)$ is denoted by $\mathcal{D}_\rho^+$.

We say that $X = (\Omega, \mathcal{A}, \mathcal{M}_t, X_t, P)$ is a *stochastic process* if $(\Omega, \mathcal{A}, P)$ is a probability space, $\{\mathcal{M}_t\}$ is an increasing system of sub-σ-fields of $\mathcal{A}$, and, for each $t \geqq 0$, X_t is an $\mathcal{M}_t$-measurable map on Ω into S. The σ-fields $\{\mathcal{M}_t\}$ are called *right-continuous* if, for each $t \geqq 0$, $\mathcal{M}_t = \mathcal{M}_{t^+} = \bigcap_{s>t} \mathcal{M}_s$. The *sample path* $X(\omega)$

of an $\omega \in \Omega$ is the map of $[0,\infty)$ into S whose value at $t \geqq 0$ is $X_t(\omega)$. Throughout the rest of this paper we consider only processes whose σ-fields $\{\mathcal{M}_t\}$ are right continuous and whose sample paths belong to D. We abuse notation by using X to denote not only the process but the map of Ω into D given by $\omega \to X(\omega)$. We shall use π to denote the measure on $\mathcal{D}$ defined by $\pi(A) = P(X^{-1}(A))$, $A \in \mathcal{D}$: that is, $\pi = P \circ X^{-1}$. We call π the *distribution* of X. A stopping time for X is a map τ of Ω into $[0,\infty]$ such that, for each $s \geqq 0$, $\{\tau \leqq s\}$ belongs to the P-completion of $\mathcal{M}_s$. We assume that each of the σ-fields $\mathcal{M}_t$ are complete with respect to P. If τ is a stopping time, then $\mathcal{M}_\tau$ is the σ-field of sets A for which $A \cap \{\tau \leqq t\} \in \mathcal{M}_t$ for each $t \geqq 0$, and X_τ is the function defined on $\{\tau < \infty\}$ by $X_\tau(\omega) = X_{\tau(\omega)}(\omega)$. It is easy to see that if each of the σ-fields $\mathcal{M}_t$ is complete with respect to P, then so is $\mathcal{M}_\tau$. The right continuity of the sample paths implies that X_τ is $\mathcal{A}$-measurable: in fact, it is $\mathcal{M}_\tau$-measurable. We set $X_\tau = \Delta$ on $\{\tau = \infty\}$. The process X is called *quasi-left-continuous* if, for any stopping time τ and any stopping times $\tau_n \uparrow \tau$ (on $\{\tau < \infty\}$), we have $X_{\tau_n} \to X_\tau$ a.s. on $\{\tau < \infty\}$.

We denote by $\mathcal{F}_\tau^+$ the sub-σ-field of $\mathcal{A}$ generated by functions $X_{\tau+t}$ as t ranges over $[0,\infty)$.

2.1. Lemma. *Let τ be a stopping time, and U an open subset of S. Let $\gamma = \inf\{t : t \geqq 0, X_{\tau+t} \in U\}$. Then γ and $X_{\tau+\gamma}$ are $\mathcal{F}_\tau^+$-measurable.*

Proof. Because X_t is right continuous in t, $\{\gamma < s\}$ is the union of the sets $\{X_{\tau+r} \in U\}$ as r ranges over the non-negative rationals less than s. But each of these sets is in $\mathcal{F}_\tau^+$, so γ is $\mathcal{F}_\tau^+$-measurable. If U is open,

$$\{X_{\tau+\gamma} \in U\} = \bigcup_n \bigcap_{m=n}^{\infty} \bigcup_k \left\{ X_{\tau+(k+1)/n} \in U, \frac{k}{m} \leqq \tau < \frac{k+1}{m} \right\}.$$

Since each set on the right belongs to $\mathcal{F}_\tau^+$, so does the set on the left. If follows that $X_{\tau+\gamma}$ is $\mathcal{F}_\tau^+$-measurable.

In [2], $\tau + \gamma$ is called the first post-γ hitting time for U, and is shown to be a hitting time for X. The proof of the lemma also establishes the analogue of Lemma 2.1 for path-defined stopping times ρ, with $\mathcal{D}_\rho^+$ in place of $\mathcal{F}_\tau^+$.

A *continuous time change* for X is a family $\{\tau_t, t \geqq 0\}$ of stopping times such that, for P-almost all ω, $\tau_t(\omega)$ is a continuous, strictly increasing function of t with $\tau_0(\omega) = 0$. It is not hard to show that if the fields $\{\mathcal{M}_t\}$ are right continuous, so are the fields $\{\mathcal{M}_{\tau_t}\}$.

2.2. Definition. The process $X = (\Omega, \mathcal{A}, \mathcal{M}_t, X_t, P)$ is called a *Markov process* if, for every stopping time τ, $E \in \Sigma$, and $t \geqq 0$,

$$P(X_{\tau+t} \in E \mid \mathcal{M}_t) = P(X_{\tau+t} \in E \mid X_\tau) \tag{2.1}$$

P-a.s. on $\{\tau < \infty\}$.

2.3. LEMMA. *Suppose $X = (\Omega, \mathcal{A}, \mathcal{M}_t, X_t, P)$ is a Markov process, and that τ is a stopping time for X. Then, if Y is a non-negative $\mathcal{F}_\tau^+$-measurable function on Ω,*

$$E(Y \mid \mathcal{M}_t) = E(Y \mid X_\tau) \tag{2.2}$$

P-a.s. on $\{\tau < \infty\}$.

PROOF. It suffices to establish (2.2) for $Y = f_1(X_{\tau+t_1}) \cdot \cdots \cdot f_n(X_{\tau+t_n})$ where n is a positive integer, $0 \leqq t_1 < \cdots < t_n$, and $f_1, \cdots, f_n$ are non-negative Σ-measurable functions on S. We do this by induction on n. For $n = 1$, it follows from (2.1) for $f = I_E$, and then for non-negative f by the usual sort of argument. Suppose (2.2) holds for such Y's for all stopping times τ provided $n = 1, \cdots, m-1$. Let $Y = (f_1(X_{\tau+t_1}))Z$, where $Z = f_2(X_{\tau+t_2}) \cdot \cdots \cdot f_m(X_{\tau+t_m})$. Then, by virtue of the induction hypothesis and well-known properties of conditional expectation,

$$\begin{aligned} E(Y \mid \mathcal{M}_\tau) &= E(E(Y \mid \mathcal{M}_{\tau+t_1}) \mid \mathcal{M}_\tau) \\ &= E(f(X_{\tau+t_1})E(Z \mid \mathcal{M}_{\tau+t_1}) \mid \mathcal{M}_\tau) \\ &= E(f(X_{\tau+t_1})E(Z \mid X_{\tau+t_1}) \mid \mathcal{M}_\tau) \\ &= E(f(X_{\tau+t_1})E(Z \mid X_{\tau+t_1}) \mid X_\tau) \\ &= E(f(X_{\tau+t_1})E(Z \mid \mathcal{M}_{\tau+t_1}) \mid X_\tau) \\ &= E(E(Y \mid \mathcal{M}_{\tau+t_1}) \mid X_\tau) \\ &= E(Y \mid X_\tau). \end{aligned}$$

This establishes (2.2) for Y's of the sort specified for all positive integers up to and including m, and so proves the theorem by induction.

2.4. COROLLARY. *Suppose that X is a Markov process, and τ a stopping time for X. Suppose that γ is a $\mathcal{F}_\tau^+$-measurable function on Ω to $[0, \infty]$. Then, for each $E \in \Sigma$,*

$$P(X_{\tau+\gamma} \in E \mid \mathcal{M}_\tau) = P(X_{\tau+\gamma} \in E \mid X_\tau) \tag{2.3}$$

P-a.s. on $\{\tau + \gamma < \infty\}$.

PROOF. Let $\gamma_n = k + 1/2^n$ on $\{k/2^n < \gamma \leqq k + 1/2^n\}$. Since these sets, on which γ_n is constant, belong to $\mathcal{F}_\tau^+$, it is clear that $X_{\tau+\gamma_n}$ is measurable with respect to $\mathcal{F}_\tau^+$. Now let $n \to \infty$. Then

$$E(f(X_{\tau+\gamma_n}) \mid \mathcal{M}_\tau) = E(f(X_{\tau+\gamma_n}) \mid X_\tau)$$

by virtue of Lemma 2.3. Now let $n \to \infty$. Then $\tau + \gamma_n \downarrow \tau + \gamma$, whence $X_{\tau+\gamma_n} \to X_{\tau+\gamma}$ by virtue of the right continuity of the process. It follows that $X_{\tau+\gamma}$ is measurable with respect to $\mathcal{F}_\tau^+$. The same is true of $I_E(X_{\tau+\gamma})$, and now (2.3) is an immediate consequence of Lemma 2.3.

If $X = (\Omega, \mathcal{A}, \mathcal{M}_t, X_t, P)$ is a stochastic processes, with paths in D as we assume throughout, so is $(D, \mathcal{D}, \mathcal{D}_{t^+}, \eta_t, \pi)$, where $\pi = P \circ X^{-1}$ and $\eta_t(f) = f(t)$ for all $f \in D$ and $t \geqq 0$. This latter process is called the *path-space process induced by X*. Since we are assuming that the σ-fields $\{\mathcal{M}_t\}$ are right continuous, an easy argument shows that, if X is Markov, so is the path-space process induced by it.

Let $(\Omega, \mathcal{F}, P)$ be a probability space, with $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ sub-σ-fields of $\mathcal{F}$. Then $\mathcal{A}$ and $\mathcal{B}$ are said to be *conditionally independent given* $\mathcal{C}$ (or $\mathcal{A} \perp \mathcal{B}$ given $\mathcal{C}$) if $P(AB \mid \mathcal{C}) = P(A \mid \mathcal{C})P(B \mid \mathcal{C})$ for each $A \in \mathcal{A}$ and $B \in \mathcal{B}$. An equivalent condition is that $P(A \mid \mathcal{B} \vee \mathcal{C}) = P(A \mid \mathcal{C})$ for each $A \in \mathcal{A}$.

2.5. Lemma. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are conditionally independent given $\mathcal{C}$. The following then hold.*

(a) *If $\mathcal{A}_0 \subset \mathcal{A}$ and $\mathcal{B}_0 \subset \mathcal{B}$, then $\mathcal{A}$ and $\mathcal{B}$ are conditionally independent given $\mathcal{A}_0 \vee \mathcal{B}_0 \vee \mathcal{C}$.*

(b) *If $\mathcal{C}_0 \subset \mathcal{C}$, then $\mathcal{A}$ is conditionally independent of $\mathcal{B} \vee \mathcal{C}_0$ given $\mathcal{C}$.*

Proof. Under the assumptions of part (a), for $A \in \mathcal{A}$, $P(A \mid \mathcal{B} \vee \mathcal{C}) = P(A \mid \mathcal{C})$. Since $\mathcal{B} \vee \mathcal{C} \supset \mathcal{B}_0 \vee \mathcal{C} \supset \mathcal{C}$, this implies that $P(A \mid \mathcal{B} \vee \mathcal{C}) = P(A \mid \mathcal{B}_0 \vee \mathcal{C})$. But $\mathcal{B} \vee \mathcal{C} = \mathcal{B} \vee (\mathcal{B}_0 \vee \mathcal{C})$, so $P(A \mid \mathcal{B} \vee (\mathcal{B}_0 \vee \mathcal{C})) = P(A \mid \mathcal{B}_0 \vee \mathcal{C})$. Thus $\mathcal{A} \perp \mathcal{B}$ given $\mathcal{B}_0 \vee \mathcal{C}$. Applying the same argument again, but reversing the roles of $\mathcal{A}$ and $\mathcal{B}$, we obtain $\mathcal{A} \perp \mathcal{B}$ given $\mathcal{A}_0 \vee \mathcal{B}_0 \vee \mathcal{C}$. This proves (a). Assume the hypotheses of (b). It suffices to show that $P(ABC \mid \mathcal{C}) = P(A \mid \mathcal{C})P(BC \mid \mathcal{C})$ for each $A \in \mathcal{A}$, $B \in \mathcal{B}$, and $C \in \mathcal{C}_0$. But then

$$E(I_A I_{BC} \mid \mathcal{C}) = I_C E(I_A \mid \mathcal{C}) I_C E(I_B \mid \mathcal{C}) = E(I_A \mid \mathcal{C}) E(I_{BC} \mid \mathcal{C}).$$

Thus (b) holds.

2.6. Theorem. *Suppose that $X = (\Omega, \mathcal{A}, \mathcal{M}_t, X_t, P)$ is a Markov process, and that τ is a stopping time for X. Then $\mathcal{M}_\tau$ and $\mathcal{F}_\tau^+$ are conditionally independent given X_τ.*

Proof. Suppose Y is $\mathcal{F}_\tau^+$-measurable. We want to show that

$$E(Y \mid \mathcal{M}_\tau) = E(Y \mid X_\tau) \qquad P\text{-a.s.} \tag{2.4}$$

This holds on $\{\tau < \infty\}$ by virtue of Lemma 2.3. Since $\{\tau = \infty\} = \{X_\tau = \Delta\}$, $\{\tau = \infty\}$ is X_τ-measurable. Also $X_{\tau+t} = X_\infty = \Delta$ for all t on $\{\tau = \infty\}$, so Y is constant on $\{\tau = \infty\}$. Thus (2.4) holds on $\{\tau = \infty\}$ as well.

If X is Markov, so is the path space process induced by X. Therefore, if ρ is a path-defined stopping time, $\mathcal{D}_\rho \perp \mathcal{D}_\rho^+$ given $f(\rho)$ (relative to the measure π on $\mathcal{D}$).

2.7. Definition. The Markov process $X = (\Omega, \mathcal{A}, \mathcal{M}_t, X_t, P)$ is said to have *stationary transition probabilities* if there is for each $t \geqq 0$, $x \in S$, and $E \in \Sigma$, a number $P_t(x, E)$ satisfying the following conditions:

1. for each t and x, $P_t(x, \cdot)$ is a probability measure on Σ,
2. for each t and E, $P_t(\cdot, E)$ is Σ-measurable,
3. for each x and E, $P_t(x, E)$ is Borel measurable in t,
4. $P_{s+t}(x, E) = \int P_s(x, dy) P_t(y, E)$,
5. $P(X_{\tau+t} \in E \mid \mathcal{M}_\tau) = P_t(X_\tau, E)$ P-a.s. on $\{\tau < \infty\}$ if τ is any stopping time for X.

Then the system $\{P_t(x, E), t \geqq 0, x \in S, E \in \Sigma\}$ is called a *transition probability function* for X.

We emphasize that throughout this paper we consider only processes with infinite lifetimes.

Suppose $X = (\Omega, \mathcal{A}, \mathcal{M}_t, X_t, P)$ is a Markov process with transition probability function $\{P_t(x, E)\}$. Then the path space process $(D, \mathcal{D}, \mathcal{D}_{t^+}, \eta_t, \pi)$ is a Markov process with $\{P_t(x, E)\}$ as its transition probability function. The distribution $\pi = P \circ X^{-1}$ can be defined in the following way. Let μ be the distribution of $X_0 : \mu(E) = P(X_0 \in E)$, $E \in \Sigma$. Then π is the measure P_μ on $\mathcal{D}$ for which

$$P_\mu(\{f : f(0) \in E_0, f(t_1) \in E_1, \cdots, f(t_n) \in E_n\}) = \int_{E_0} \mu(dx_0) \int_{E_1} P_{t_1}(x_0, dx_1) \int_{E_2} P_{t_2 - t_1}(x_1, dx_2) \cdots \cdots \int_{E_{n-1}} P_{t_{n-2} - t_{n-1}}(x_{n-2}, dx_{n-1}) P_{t_n - t_{n-1}}, E_n \tag{2.5}$$

for each $0 < t_1 < \cdots < t_n < \infty$ and $E_0, E_1, \cdots, E_n$ in Σ. The measure P_μ defined on $\mathcal{D}$ by (2.5) is said to be determined by the initial measure μ. It looks as if for any probability measure μ, the equations (2.5) define a measure P_μ on $(D, \mathcal{D})$. What we have assumed so far, however, does not guarantee the existence of such a measure. If, however, we replace $(D, \mathcal{D})$ by $(S^\infty, \Sigma^\infty)$, where S^∞ is the product space $S^{[0,\infty)}$ and Σ^∞ the coordinate σ-field over Σ^∞, then Kolmogorov's consistency theorem applies to yield a measure P_μ on $(S^\infty, \Sigma^\infty)$ for which (2.5) holds (see, for example, [1], p. 17). Assume this has been done. For each x, let $P_x = P_\mu$, where $\mu = \delta_x$. Now, if μ is the initial measure corresponding to π, P_μ assigns outer measure 1 to D, and so we can define P_μ on $(D, \mathscr{D})$ (obtaining π). Our assumptions, however, do not imply that the various P_x's assign outer measure 1 to D. Suppose, however, that they do, and that accordingly each P_x defines a probability measure on $(D, \mathscr{D})$. Let θ_t be the *shift* on D defined by $\theta_t f(s) = f(s+t)$ for s and t in $[0,\infty)$. Then $(D, \mathscr{D}, \mathscr{D}_t, \eta_t, \theta_t, P_x)$ defines a Markov process in the sense of Blumenthal and Getoor. However, this Markov process need not have the regularity properties possessed by the original process X, which we require for a complete proof of the main theorem. (For example, the process need not be strong Markov, and replacement of $\mathscr{D}_t$ by $\mathscr{D}_{t^+}$ may destroy the Markov property.) It turns out to be necessary for some purposes to assume something even stronger, namely that $(D, \mathscr{D}, \mathscr{D}_{t^+}, \eta_t, \theta_t, P_x)$ be a standard process in the sense of Blumenthal and Getoor ([1], p. 45). If this holds, we say that X *has a transition function which induces a standard process on D.*

2.8. THEOREM (Dynkin). *Suppose $(\Omega, \mathscr{A}, \mathscr{M}_t, X_t, \theta_t, P_x)$ is a standard Markov process (in the sense of Blumenthal and Getoor) with a transition function $\{P_t(x,E)\}$ which induces a standard process on D. Let f be a bounded continuous real-valued function on S. Then, for each $t \geqq 0$, $(P_t f)(X_s)$ is P_μ-almost surely right continuous in s.*

PROOF. This is an immediate consequence of theorem 4.10 and 4.11 on page 125 of [4].

We next state for reference and prove two facts about S-valued functions.

2.9. PROPOSITION. *Let $I = [a,b]$ be a finite closed subinterval of $[0,\infty)$, and f a function on I into S which is right continuous on $[a,b]$ and has left limits throughout $(a,b]$. Let $\delta > 0$, and $\varepsilon_m \downarrow 0$. Suppose that, for each $m = 1, 2, \cdots$, there is a t_m with $[t_m, t_m + \delta] \subset I$ for which $d(f(t), f(t_m)) \leqq \varepsilon_m$ for all $t \in [t_m, t_m + \delta]$. Then there is a subinterval of I on which f is constant.*

PROOF. Assume the hypotheses. There is a subsequence of $\{t_m\}$ converging to some t_0 for which $[t_0, t_0 + \delta] \subset I$. We may assume without loss of generality that $t_m \to t_0$, in fact, that either $t_m \uparrow t_0$ or $t_m \downarrow t_0$. Suppose $t_m \downarrow t_0$. Let $t \in (t_0, t_0 + \delta)$. For sufficiently large m, $t_0 \leqq t_m < t < t_m + \delta$. We have

$$d(f(t), f(t_0)) \leqq d(f(t_0), f(t_m)) + d(f(t_m), f(t)).$$

As $m \to \infty$, $d(f(t_0), f(t_m)) \to 0$ by right continuity, while $d(f(t_m), f(t)) \leqq \varepsilon_m < 0$. Thus $d(f(t), f(t_0)) = 0$, so $f(t) = f(t_0)$. This shows that f is constant on $[t_0, t_0 + \delta)$. Suppose that $t_m \uparrow t_0$. Let $t \in (t_0, t_0 + \delta/2)$. For sufficiently large m, $t_m \leqq t_0 < t < t_0 + \delta$, and

$$d(f(t), f(t_0)) \leqq d(f(t_m), f(t_0)) + d(f(t_m), f(t)).$$

Since both $d(f(t_m), f(t_0)) \leqq \varepsilon_m$ and $d(f(t_m), f(t)) \leqq \varepsilon_m$, we again conclude that $f(t) = f(t_0)$. This shows that f is constant on $[t_0, t_0 + \delta/2)$, completing the proof.

2.10. PROPOSITION. *Suppose f and g are right continuous S-valued functions on $[0, \infty)$ without intervals of constancy. Suppose σ_1 and σ_2 are continuous, strictly increasing maps of $[0, \infty)$ onto itself for which $f \circ \sigma_1 = f \circ \sigma_2 = g$. Then $\sigma_1 = \sigma_2$.*

PROOF. Assume the hypotheses. Then $f \circ \sigma_1 \circ \sigma_2^{-1} = f$, and $\sigma = \sigma_1 \circ \sigma_2^{-1}$ is a strictly increasing map of $[0, \infty)$ onto itself. We must show that σ is the identity map. Suppose not. Then there is an $s \in [0, \infty)$ for which $\sigma(s) \neq s$. Either $\sigma(s) < s$ or $\sigma(s) > s$. Suppose $\sigma(s) < s$. Let $t \in (\sigma(s), s)$. We have $\sigma^{n+1}(s) < \sigma^n(t) < \sigma^n(s)$, where σ^n denotes the nth iterate of σ. If $u = \lim_n \sigma^n(s)$, then $u = \lim_n \sigma^n(t)$. By right continuity, and $f \circ \sigma = f$, $f(u) = f(s)$ and $f(u) = f(t)$. Thus f is constant on $(\sigma(s), s)$, contradicting the assumption that it has no interval of constancy. Since $\sigma(s) < s$ leads to a contradiction, we must have $\sigma(s) > s$. But then $s > \sigma^{-1}(s)$; since $f \circ \sigma^{-1} = f$, this also leads to a contradiction. Therefore $\sigma(s) = s$, completing the proof of (b).

In [2], we state a theorem on the disintegration of measures. We restate it here in the form in which we use it. First, suppose that $(D, \mathcal{D}, \pi)$ is a probability space. Let $\mathcal{E}$ be a countably generated sub-σ-field of $\mathcal{D}$. The fibers of $\mathcal{E}$ are then $\mathcal{E}$-measurable. Let $\mathcal{F}$ be the class of all fibers of $\mathcal{E}$. There is an obvious correspondence between the σ-field of all subsets of $\mathcal{F}$ whose unions are members of $\mathcal{D}$ and the sub-σ-field of $\mathcal{D}$ consisting of such unions. (Note that $\mathcal{E}$ is contained in said sub-σ-field.) This correspondence takes the restriction of π to that subfield into a measure on the σ-field of subsets of $\mathcal{F}$ to which the subfield of $\mathcal{D}$ corresponds. We denote the measure by π, thus at the same time conserving and abusing notation.

2.11. THEOREM (Disintegration of measures). *Let D be a complete separable metric space, with $\mathcal{D}$ its class of Borel sets. Suppose that π is a probability measure on $\mathcal{D}$, and that $\mathcal{E}$ is a countably generated sub-σ-field of $\mathcal{D}$, with $\mathcal{F}$ its class of fibers. Then there is a family $\{\pi_\xi\}_{\xi \in \mathcal{F}}$ of measures on $\mathcal{D}$ satisfying the following conditions ($\xi(x)$ denotes the member of $\mathcal{F}$ to which x belongs):*

(i) *for each* $A \in \mathscr{D}$, $\pi_{\xi(x)}(A)$ *is an* $\mathscr{E}$*-measurable function of* x,

(ii) *for each* $A \in \mathscr{D}$ *and* $E \in \mathscr{E}$,

$$\pi(A \cap E) = \int_E \pi_\xi(A)\pi(d\xi),$$

and

(iii) *for* π*-almost all* $\xi \in \mathscr{F}$, π_ξ *is a probability measure with* $\pi_\xi(\xi) = 1$.

If Z *is a bounded* $\mathscr{D}$*-measurable real-valued function on* D, *then* $x \to \int Z d\pi_{\xi(x)}$ *is an* $\mathscr{E}$*-measurable version of* $E_\pi(Z \mid \mathscr{E})$.

2.12. Corollary. *Let* Z *be a bounded* $\mathscr{D}$*-measurable real-valued function on* D, *and* $\mathscr{E}$ *a countably generated sub-*σ*-field of* $\mathscr{D}$. *Then* Z *is* π*-a.s.* $\mathscr{E}$*-measurable if and only if* Z *is* π_ξ*-almost surely constant on* π*-almost all fibers of* $\mathscr{E}$.

Proof. Suppose Z is π-almost surely equal to an $\mathscr{E}$-measurable function Y. Then Y is constant on fibers of $\mathscr{E}$. Since $A = [Y \neq Z]$ is $\mathscr{D}$-measurable and π-null, its intersection with π-almost all fibers ξ of $\mathscr{E}$ has π_ξ-measure 0, for $0 = \pi(A) = \int \pi_\xi(A)\pi(d\xi) = \int \pi_\xi(A \cap \xi)\pi(d\xi)$. But if $\xi \subset A^c$, Z on ξ is equal to the constant value of Y on ξ.

Suppose that Z is π_ξ-almost surely constant on π-almost all fibers of $\mathscr{E}$. Of these fibers those for which π_ξ is a probability measure concentrating on ξ have the property that $\int Z d\pi_\xi$ is the π_ξ-a.s. constant value of Z on ξ. Therefore, for π-almost all ξ, $Z = \int Z d\pi_\xi$ on ξ. Since the function equal to $\int Z d\pi_\xi$ on ξ is $\mathscr{E}$-measurable by (i), Z is π-a.s. $\mathscr{E}$-measurable. This completes the proof of the corollary.

We shall also use the fact that, under the hypotheses of the corollary, Z is $\mathscr{E}$-measurable if and only if it is constant on fibers of $\mathscr{E}$. This follows easily from the case where Z is the indicator of an $\mathscr{A}$-measurable set, which in turn follows from *Blackwell's Theorem*, stated on page 38 of [6].

Suppose that $\{\pi_\xi\}_{\xi \in \mathscr{F}}$ is a disintegration of measures with respect to $\mathscr{E}$, and that the countably generated σ-fields $\mathscr{A}$ and $\mathscr{B}$ are conditionally independent (relative to π) given $\mathscr{E}$. Then, for π-almost all $\xi \in \mathscr{F}$, A and $\mathscr{B}$ are independent relative to π_ξ. This is an almost immediate consequence of the fact that for any $C \in \mathscr{D}$, $\pi_{\xi(x)}(C)$, as a function of x, is a version of the conditional probability $\pi(\mathscr{C} \mid \mathscr{E})$.

Suppose that $(D, \mathscr{D}, \mathscr{D}_{t^+}, \eta_t, \theta_t, P_x)$ is a Markov process in the sense of Blumenthal and Getoor, which is normal in their sense ([1], page 30), that is, $P_x(\eta_0 = x)$ for all $x \in S$. Let $\mathscr{E}$ be a countably generated sub-σ-field of $\mathscr{D}$, with $\mathscr{F}$ its class of fibers. Then, for each $x \in S$, there is a disintegration $\{P_{x,\xi}\}_{\xi \in \mathscr{F}}$ of the measure P_x. We shall require the fact that it is possible to arrange things so that $P_{x,\xi}(A)$ is jointly measurable in (x, ξ) for each $A \in \mathcal{D}$.

2.13. PROPOSITION. *Suppose that* $(D, \mathcal{D}, \mathcal{D}_{\tau^+}, \eta_\tau, \theta_\tau, P_x)$ *is a Markov process in the sense of Blumenthal and Getoor. Let* $\mathcal{E}$ *be a countably generated sub-*σ*-field of* $\mathcal{D}$, *and* $\mathcal{F}$ *the class of fibres of* $\mathcal{E}$. *Then there is a system* $\{P_{x,\xi}(A)\}$, *with* x *ranging over* S, ξ *over* $\mathcal{F}$, *and* A *over* $\mathcal{D}$ *such that, for each* $x \in S$, $\{P_{x,\xi}\}_{\xi\in\mathcal{F}}$ *is a disintegration of* P_x *relative to* $\mathcal{E}$, *and, for each* $A \in \mathcal{D}$, $P_{x,\xi}(A)$ *is* $\Sigma \times \mathcal{E}$*-measurable in* (x, ξ).

PROOF. This proposition is an immediate consequence of a result on the existence of simultaneous disintegrations of measures depending measurably on a parameter. This result was first shown to us by Neil Falkner, who later observed that it is stated on page 90 of [7].

§3. The times ρ_{nk}, and some sub-fields of $\mathcal{D}$

For each path-defined stopping time ρ on D and $r > 0$, let $T(\rho, r, f) = \inf\{t : t \geqq \rho(f), d(f(t), f(\rho)) \geqq r\}$. $T(\rho, r) = T(\rho, r, \cdot)$ is a path-defined stopping time. We define $\mathcal{T}$ to be the smallest class of path-defined stopping times with the following two properties:

(i) $0 \in \mathcal{T}$,

(ii) if $\rho \in \mathcal{T}$, and r is a rational in $(0, 1]$, then $T(\rho, r) \in \mathcal{T}$.

Clearly, $\mathcal{T}$ is countable. We define D_0 to be the set of all $f \in \mathcal{D}$ for which $T(\rho, r, f) = \lim_n T(\rho, r - (1/n), f)$ for all $\rho \in \mathcal{T}$. It is clear that $D_0 \in \mathcal{D}$.

3.1. LEMMA. *If* $X = (\Omega, \mathcal{A}, \mathcal{M}_t, X_t, P)$ *is a quasi-left-continuous process, then* $\pi(D_0) = 1$.

PROOF. It suffices to show that if ρ is a path-defined stopping time $r > 0$, and X is *qlc*, then $T(\rho, r) = \lim_n T(\rho, r - (1/n))$ π-a.s. Clearly $T(\rho, r - 1/n) \leqq T(\rho, r)$, and $T(\rho, r - 1/n)$ increases with n. Let $T_\infty = \lim_n T(\rho, r - 1/n)$: then $T_\infty \leqq T(\rho, r)$. If $f \in D$, and $\rho(f) < \infty$, $d(f(T(\rho, r - 1/n)), f(\rho)) \geqq r - 1/n$ by virtue of the right continuity of f. By virtue of quasi-left-continuity, $f(T(\rho, r - 1/n)) \to f(T_\infty)$ π-a.s., whence $d(f(T_\infty), f(\rho)) \geqq r$ π-a.s. on $\{T_\infty < \infty\}$. Thus $T(\rho, r) \leqq T_\infty$ π-a.s., so $T(\rho, t) = T_\infty$ π-a.s., proving the lemma.

Let $\varepsilon_n = 1/2^n$, $n = 1, 2, \cdots$. For each such n, the sequence $\{\rho_{nk}\}_{k=0}^\infty$ is defined inductively by

(i) $$\rho_{n0} = 0,$$

(ii) $$\rho_{n,k+1} = T(\rho_{nk}, \varepsilon_n).$$

Clearly $\{\rho_{nk} : n = 1,2,\cdots, k = 0,1,\cdots\} \subset \mathscr{T}$. For each n, let $k_n = \sup\{k : \rho_{nk} < \infty\}$. If $f \in D$, and $k_n(f) < \infty$, then $\rho_{nk}(f) < \infty$ for $k \leqq k_n$, while $\rho_{nk}(f) = \infty$ for $k > k_n$; if $k_n(f) = \infty$, $\rho_{nk}(f) < \infty$ for each k. We list some properties of the ρ_{nk}'s in the following lemma.

3.2. LEMMA. *Suppose $f \in D$. Then the following hold:*

(a) $\{\rho_{nk}\}_{k=1}^{k_n(f)}$ *is strictly increasing,*

(b) *if $n > m$, and $\rho_{n,k+1}(f) < \infty$, then there is at most one value of i for which $\rho_{nk}(f) < \rho_{mi}(f) < \rho_{n,k+1}(f)$,*

(c) $\{\rho_{nk}(f)\}_{k=1}^{\infty}$ *has no finite limit points,*

(d) *if f has no intervals of constancy, $\{\rho_{nk}(f) : n = 1,2,\cdots, k = 0,1,\cdots\}$ is dense in $[0,\infty)$.*

PROOF. Property (a) is obvious. Property (b) is a consequence of the definitions of ε_n, ρ_{nk}, and the triangle inequality. Because f is right continuous,

$$d(f(\rho_{nk}), f(\rho_{n,k+1})) \geqq \varepsilon_n \qquad \text{if } \rho_{n,k+1}(f) < \infty,$$

and since f has right and left limits everywhere, (c) follows. Suppose there is no $\rho_{nk}(f)$ in (a,b). Let $l_n = \sup\{l : \rho_{nl}(f) \leqq a\}$. Then $d(f(t), f(\rho_{n,l_n})) < \varepsilon_n$ for $t \in (a,b)$, and it follows that $d(f(s), f(t)) < 2\varepsilon_n$ for each s, t in (a,b). If this holds for each n, f must be constant on (a,b). This proves (d).

Let $\mathscr{C}$ be the sub-σ-field of $\mathscr{D}$ generated by the maps $f \to f(\rho)$, $\rho \in \mathscr{T}$. Then $\mathscr{C}$ is countably generated. If f and g are in the same fiber of $\mathscr{C}$, we write $f \equiv g \pmod{\mathscr{C}}$. Clearly $f \equiv g \pmod{\mathscr{C}}$ iff $f(\rho(f)) = g(\rho(g))$ for each $\rho \in \mathscr{T}$.

3.3. LEMMA. *Suppose f and g are members of D_0, that neither has an interval of constancy, and that $f \equiv g \pmod{\mathscr{C}}$. Suppose $m < n$, and that $\rho_{n,k+1}(f) < \infty$. Then $\rho_{nk}(f) < \rho_{mi}(f) < \rho_{n,k+1}(f)$ if and only if $\rho_{nk}(g) < \rho_{mi}(g) < \rho_{n,k+1}(g)$.*

PROOF. Assume the hypotheses of the lemma. Because of Lemma 3.2(b) it is enough to prove the following: if there is no i for which $\rho_{nk}(f) < \rho_{mi}(f) < \rho_{n,k+1}(f)$, then there is no i for which $\rho_{nk}(g) < \rho_{mi}(g) < \rho_{n,k+1}(g)$. An easy inductive argument shows that this in turn is demonstrated if we can show the following. Suppose that i is the largest index for which $\rho_{mi}(f) \leqq \rho_{nk}(f)$ and also the largest index for which $\rho_{mi}(g) \leqq \rho_{nk}(g)$. Then $\rho_{m,i+1}(f) > \rho_{n,k+1}(f)$ implies that $\rho_{m,i+1}(g) > \rho_{n,k+1}(g)$. Assume, then, that i is the largest index for which $\rho_{mi}(f) \leqq \rho_{nk}(f)$ and the largest index for which $\rho_{mi}(g) \leqq \rho_{nk}(g)$, and that $\rho_{m,i+1}(f) > \rho_{n,k+1}(f)$. Let V be the open sphere of radius ε_m and center $x_{mi} = f(\rho_{mi})(= g(\rho_{mi}))$. The f-image of $[\rho_{nk}(f), \rho_{n,k+1}(f)]$ is contained in V. In fact,

$$\text{(a)} \qquad d(f[\rho_{nk}(f), \rho_{n,k+1}(f)], V^c) > 0.$$

To prove (a), suppose it does not hold. Then there are points $\{t_l\} \subset [\rho_{nk}(f), \rho_{n,k+1}(f)]$ with $d(f(t_l), V^c) \to 0$ as $l \to \infty$. Since $[\rho_{n,k}(f), \rho_{n,k+1}(f)] \subset [\rho_{mi}(f), \rho_{n,k+1}(f)]$, this clearly implies that for each j, $T(\rho_{mi}, \varepsilon_m - (1/j), f) \leqq \rho_{n,k+1}(f)$. But $f \in D_0$, so

$$T(\rho_{mi}, \varepsilon_m - (1/j), f) \to T(\rho_{mi}, \varepsilon_m, f) \qquad \text{as } j \to \infty,$$

yielding $\rho_{m,i+1}(f) = T(\rho_{mi}, \varepsilon_m, f) \leqq \rho_{n,k+1}(f)$, contrary to assumption.

Let $x_{nk} = f(\rho_{nk})(= g(\rho_{nk}))$. For each $r \geqq 0$, let $T_0(r) = \rho_{nk}$, $T_1(r) = T(T_0(r), r), \cdots$. Call $t_i(r) = T_i(r, f)$ an *r-time for f*, and $\tilde{t}_i(r) = T_i(r, g)$ an *r-time for g*, $i = 0, 1, \cdots$. Let $y_i(r) = f(t_i(r))$ $(= g(\tilde{t}_i(r))$, $i = 0, 1, \cdots$. The following is an easy consequence of the definitions involved.

(b) $t_N(r)$ is the largest r-time for f less than $\rho_{n,k+1}(f)$ iff $d(y_i(r), x_{nk}) < \varepsilon_n$, $i = 1, \cdots, N$ and $d(y_{N+1}(r), x_{nk}) \geqq \varepsilon_n$.

It follows from (b) that if $t_N(r)$ is the largest r-time for f less than $\rho_{n,k+1}(f)$, then $\tilde{t}_N(r)$ is the largest r-time for g less than $\rho_{n,k+1}(g)$.

Let $d(f[\rho_{nk}(f), \rho_{n,k+1}(f)], V^c) > \delta > 0$, and r a rational, $0 < r < \delta$. For each $i = 1, \cdots, N$, let W_i be the sphere of radius r and center $y_i(r)$. Let $W = W_1 \cup \cdots \cup W_N$. It is easy to see that W covers both the f-image of $[t_0(r), t_N(r)]$ and the g-image of $[\tilde{t}_0(r), \tilde{t}_N(r)]$. Since $d(W, V^c) > \delta$, it follows that the g-image of $[\tilde{t}_0(r), \tilde{t}_N(r)]$ is contained in V, whence $\rho_{m,i+1}(f) \geqq t_N(r)$. Now let $r \downarrow 0$ through rationals. $t_N(t)$ is the largest r-time for f less than $\rho_{n,k+1}(f)$, so $\tilde{t}_N(r)$ is the largest r-time for g less than $\rho_{n,k+1}(g)$. The argument used to establish 3.1(d) shows that $\{\tilde{t}_N(r)\}$ has $\rho_{n,k+1}(g)$ as a limit point as $r \to 0$. It follows that $\rho_{m,i+1}(g) \geqq \rho_{n,k+1}(g)$. If $\rho_{m,j+1}(g) = \rho_{n,k+1}(g)$, then

$$d(f(\rho_{n,k+1}), x_{mi}) = d(g(\rho_{n,k+1}), x_{mi}) \geqq \varepsilon_m,$$

contradicting the assumption that $\rho_{m,i+1}(f) > \rho_{n,k+1}(f)$. Therefore $\rho_{m,i+1}(g) > \rho_{n,k+1}(g)$. This completes the proof of the lemma.

The ρ_{nk}'s are not constant on fibers of $\mathcal{O}$. The lemma shows, however, that the ordering of the $\rho_{nk}(f)$'s as n and k vary is fixed by specifying the fiber to which f belongs — provided we restrict ourselves to functions f which have no intervals of constancy, and which also satisfy a somewhat artifical condition whose usefulness is a consequence of the quasi-left-continuity of the process. It would be of interest to know whether the ordering of the $\rho_{nk}(f)$'s is determined by the fiber to which f belongs if these conditions are not satisfied.

Let D_1 be the set of all $f \in D_0$ having no intervals of constancy. We say that functions f and g in D differ by *a change of variable* if there is a continuous, strictly increasing map σ of $[0, \infty)$ onto itself for which $g = f \circ \sigma$. We call such a function σ *a change of variable which takes f into g*. Such a change of variable is unique by virtue of Proposition 2.10.

3.4. LEMMA. *Suppose f and g differ by a change of variable σ, for which $g = f \circ \sigma$. Then, for each n and k,*

(a) $\rho_{nk}(f) = \sigma(\rho_{nk}(g))$,

and

(b) $\sigma(s + \rho_{nk}(\theta_s g)) = \sigma(s) + \rho_{nk}(\theta_{\sigma_s} f)$, *for each* $s \in [0, \infty)$.

PROOF. We prove (a) for each n by induction on k. For $k = 0$, (a) is obvious. Suppose (a) is true as it stands. Then

$$\begin{aligned}\sigma(\rho_{n,k+1}(g)) &= \sigma(\{\inf\{t : t \geqq \rho_{nk}(g), d(g(t), g(\rho_{nk})) \geqq \varepsilon_n\}) \\ &= \inf\{\sigma(t) : \sigma(t) \geqq \sigma(\rho_{nk}(g)), d(f(\sigma(t)), f(\sigma(\rho_{nk}(g)) \geqq \varepsilon_n\} \\ &= \inf\{s : s \geqq \rho_{nk}(f), d(f(s), f(\rho_{nk})) \geqq \varepsilon\} \\ &= \rho_{n,k+1}(f),\end{aligned}$$

which verifies the induction step.

We now prove (b). Fix s. Let f^* and g^* be defined by

$$f^*(t) = f(t + \sigma(s)) - f(\sigma(s)), \quad g^*(t) = g(t + s) - g(s), \qquad t \in [0, \infty).$$

Then

$$f^*(\sigma(t + s) - \sigma(s)) = f(\sigma(t + s)) - f(\sigma(s)) = g^*(t).$$

Thus if σ^* is defined by $\sigma^*(t) = \sigma(t + s) - \sigma(s)$, $t \in [0, \infty)$, σ^* is a change of variable for which $f^* \circ \sigma^* = g^*$. It now follows from (a) that $\rho_{nk}(f^*) = \sigma^*(\rho_{nk}(g^*))$. But f^* and g^* differ by constants from $\theta_{\sigma_s} f$ and $\theta_s g$ respectively, so $\rho_{nk}(\theta_{\sigma_s} f) = \rho_{nk}(f^*)$, and $\rho_{nk}(\theta_s g) = \rho_{nk}(g^*)$. Therefore

$$\rho_{nk}(\theta_{\sigma_s} f) = \sigma^*(\rho_{nk}(\theta_s g)) = \sigma(s + \rho_{nk}(\theta_s g)) - \sigma(s).$$

This completes the proof of (b).

3.5. THEOREM. *Suppose that f and g belong to D_1. Then $f \equiv g \pmod{\mathcal{O}}$ if and only if f and g differ by a change of variable.*

PROOF. It is easy to see that if f and g determine the same trajectory, then $f \equiv g \pmod{\mathcal{O}}$. To prove the converse, begin by assuming that $f \equiv g \pmod{\mathcal{O}}$. Let $\sigma_0 = \{(\rho_{nk}(f), \rho_{nk}(g)) : n = 1, 2, \cdots, k = 0, 1, \cdots\}$: that is, $\sigma_0(\rho_{nk}(g)) = \rho_{nk}(f)$.

(1) σ_0 *is a function*

This requires that $\rho_{ml}(g) = \rho_{nk}(g) \Rightarrow \rho_{ml}(f) = \rho_{nk}(f)$, which follows from Lemma 3.3.

(2) *σ_0 is strictly increasing and unbounded*

This is a consequence of 3.2(a) and (d).

(3) *σ_0 is right continuous on its domain*

For suppose $\rho_{n_m,k_m}(g) \downarrow \rho_{l,j}(g)$ as $m \to \infty$. We now make repeated use of the ordering property asserted by Lemma 3.3. First, $\{\rho_{n_m,k_m}(f)\}$ is non-increasing since $\{\rho_{n_m,k_m}(g)\}$ is. Each term of $\{\rho_{n_m,k_m}(f)\}$ is bounded below by $\rho_{l,j}(f)$: let $a = \lim_m \rho_{n_m,k_m}(f)$. Let $\varepsilon > 0$. Because of the density of the $\rho_{m,n}(f)$'s in $[0,\infty)$, there is an (n_0, k_0) for which $\rho_{l,j}(f) < \rho_{n_0,k_0}(f) < \rho_{l,j}(f) + \delta$. Then $\rho_{l,j}(g) < \rho_{n_0,k_0}(g)$, so $\rho_{n_m,k_m}(g) < \rho_{n_0,k_0}(g)$ for all sufficiently large values of m. It follows that $\rho_{n_m,k_m}(f) < \rho_{n_0,k_0}(f)$ for such values of m. Therefore $a < \rho_{n_0,k_0}(f) < \rho_{l,j}(f) + \delta$. Since δ is an arbitrary positive number, $a \leqq \rho_{l,j}(f)$, so $a = \rho_{l,j}(f)$. Therefore $\rho_{n_m,k_m}(f) \downarrow \rho_{l,j}(g)$ implies $\rho_{n_m,k_m}(f) \downarrow \rho_{l,j}(f)$. Since σ_0 is increasing on its domain, this suffices to establish its right continuity.

(4) *$f \circ \sigma_0 = g$ on the domain of σ_0*

For $f(\sigma_0(\rho_{nk}(g))) = f(\rho_{nk}(f)) = g(\rho_{nk}(g))$.

Now extend σ to all of $[0,\infty)$ by right continuity: $\sigma(t) = \lim_m \sigma_0(\rho_{n_m,k_m}(g))$, where $t < \rho_{n_m,k_m}(g)$ t as $m \to \infty$. The limit exists and is independent of the particular sequence $\{\rho_{n_m,k_m}(g)\}$ chosen because of the monotonicity of σ_0, so σ is well-defined. σ is an extension of σ_0 because of the right continuity of σ_0.

(5) *σ is strictly increasing and unbounded*

This follows immediately from the corresponding properties of σ_0

(6) *σ is right continuous*

Suppose $\{t_n\}$ is a strictly decreasing sequence with limit t. There is a sequence $\{(n_m, k_m)\}$ such that (i) $\rho_{n_m,k_m}(g)$ is strictly decreasing in m and has limit t as $m \to \infty$, and (ii) $|\sigma(t_m) - \rho_{n_m,k_m}(f)| \to 0$ as $m \to \infty$. Then $\{\sigma(t_m)\}$ and $\{\rho_{n_m,k_m}(f)\}$ have the same limit points. Since the latter sequence converges to $\sigma(t)$, so does the former.

(7) *σ is continuous*

Suppose $\sigma(t) - \sigma(t-0) > \delta > 0$. For each n, let $k_n = \sup\{k : \rho_{nk}(g) \leqq t\}$. Then $\sigma(\rho_{n,k_n}(g)) \leqq \sigma(t) < \sigma(\rho_{n,k_n+1}(g))$, so $\sigma(\rho_{n,k_n+1}(g)) - \sigma(\rho_{n,k_n}(g)) > \delta$, that is: $\rho_{n,k_n+1}(f) - \rho_{n,k_n}(f) > \delta$. Thus, for each n, there is an interval I_n containing t with $|I_n| > \delta$ and such that the oscillation of f on I_n is no greater than $2\varepsilon_n$. (This last fact follows from the definition of the ρ_{ni}'s and the triangle inequality.) Taking a subsequence of n's for which the endpoints of I_n converge, we produce a non-degenerate interval on which f is constant. This contradicts the assumption that f has no intervals of constancy. Thus σ is left-continuous. Since we already know that σ is right-continuous, (7) is established.

(8) $f \circ \sigma = g$

This follows from the corresponding property of σ_0, and the right continuity of f and g.

3.6. LEMMA. *Suppose τ is a path-defined stopping time relative to $\{\mathcal{D}_{t^+}\}$. Then $\theta_\tau^{-1}(\mathcal{O})$ is the sub-σ-field of $\mathcal{D}$ generated by the maps $f \to f(\tau + \rho \circ \theta_\tau)$ as ρ ranges over $\mathcal{T}$, and $\mathcal{O} \subseteq \mathcal{D}_\tau \vee \theta_\tau^{-1}(\mathcal{O})$.*

PROOF. If f is replaced by $\theta_\tau f$, $f(\rho) = f(\rho(f))$ becomes $(\theta_\tau f)(\rho(\theta_\tau f)) = f(\tau(f) + \rho(\theta_\tau f))$, so the first statement is obvious. Let $\mathcal{P}$ be the σ-field generated by sets of the form $\{\tau \leqq s\}$ and $\{\rho \leqq \tau,\ \rho \leqq s,\ f(\rho) \in E\}$ as s and t range over $[0,\infty)$, ρ over $\mathcal{T}$, and E over Σ. Then $\mathcal{P}$ is a countably generated sub-σ-field of $\mathcal{D}_\tau$, and $\mathcal{P} \vee \theta_\tau^{-1}(\mathcal{O})$ is countably generated. Suppose f and g belong to the same fiber ξ of $\mathcal{P} \vee \theta_\tau^{-1}(\mathcal{O})$. Since they are in the same fiber of $\mathcal{P}$, $\tau(f) = \tau(g)$, and, for each $\rho \in \mathcal{T}$, $\rho(f) \leqq \tau(f)$ if and only if $\rho(g) \leqq \tau(g) (= \tau(f))$, and then $\rho(f) = \rho(g)$ and $f(\rho(f)) = g(\rho(g))$. It follows from 3.2(d) that the restrictions of f and g to $[0, \tau(f)]$ coincide. Let $\rho^*_{nk}(f) = \tau(f) + \rho_{nk}(\theta_\tau f)$, $\rho^*_{nk}(g) = \tau(g) + \rho_{nk}(\theta_\tau g)$. Since $\tau(f) = \tau(g)$, and since f and g are in the same fiber of $\theta_\tau^{-1}(\mathcal{O})$, $f(\rho^*_{nk}(f)) = g(\rho^*_{nk}(g))$ for each (n, k). Let σ^*_0 be the set of all ordered pairs $(\rho^*_{nk}(f), \rho^*_{nk}(g))$. Then, as in the proof of Theorem 3.5, σ^*_0 is a function defined on a dense subset of $[\tau(f), \infty)$ which extends to a strictly increasing, continuous map σ^* of $[\tau(f), \infty)$ onto itself. If we let f^* and g^* be the restriction to $[\tau(f), \infty)$ of f and g respectively, it follows, again as in the proof of Theorem 3.5, that $f^* \circ \sigma^* = g^*$. If we let σ be the identity map on $[0, \tau(f))$, and equal to σ^* on $[\tau(f), \infty)$, σ is a change of variable which takes f into g. If follows from Theorem 3.5 that f and g are in the same fiber of $\mathcal{O}$. Thus every fiber of $\mathcal{O}$ is a union of $\mathcal{D}_\tau \vee \theta_\tau^{-1}(\mathcal{O})$. Blackwell's theorem now implies that $\mathcal{O} \subset \mathcal{D} \vee \theta_\tau^{-1}(\mathcal{O})$.

§4. A fundamental property of Markov processes

In this section, we assume that $X = (\Omega, \mathcal{A}, \mathcal{M}_t, X_t, P)$ is a quasi-left-continuous Markov process whose sample paths have no interval of constancy, and that π is its distribution. For each $\rho \in \mathcal{T}$, we denote by $\mathcal{O}_\rho$ the sub-σ-field of $\mathcal{D}$ generated by the sets $\{f : \gamma(f) \leqq \rho(f) < \infty, f(\gamma) \in E\}$ as γ ranges over $\mathcal{T}$ and E over Σ. If $\rho = \rho_{mi}$, $\mathcal{O}_\rho$ is denoted by $\mathcal{O}_{mi}$. Suppose f and g are members of D_1 belonging to the same fiber of $\mathcal{O}_\rho$. The argument used to prove Lemma 3.3 shows that $\gamma_1(f) \leqq \gamma_2(f) \leqq \rho(f)$ iff $\gamma_1(g) \leqq \gamma_2(g) \leqq \rho(g)$ for all γ_1 and γ_2 in $\mathcal{T}$. In other words, the sub-class of $\mathcal{T}$ consisting of those γ's for which $\gamma(f) \leqq \rho(f)$, as well as the relative order of their values $\gamma(f)$, is fixed on fibers of $\mathcal{O}_\rho$. This will be referred to as "the ordering property". It holds only with the qualification that f range over members of D_1, but, since $\pi(D_1) = 1$ by virtue of Lemma 3.1 and the assumption of no intervals of constancy, we can and will assume for the rest of this section that the members of D to which we refer are in fact members of D_1. Then Lemma 3.3 (the ordering property on the fibers of $\mathcal{O}$) implies that $\mathcal{O}_\rho \subset \mathcal{O}$. Since $\sup\{\rho_{mi}(f) : m = 1, 2, \cdots, i = 0, 1, \cdots\} = \infty$ by 3.2(d), it follows that $\mathcal{O} = \vee \mathcal{O}_{mi}$.

Fix m and i. Let $\mathcal{F}$ be the class of all fibers of $\mathcal{O}_{ni}$. The subsets of $\mathcal{F}$ whose unions belong to $\mathcal{D}$ form a σ-field on which π induces a measure in the obvious way, and we abuse notation by using π to denote this measure as well. Since $\mathcal{O}_{mi}$ is countably generated, there is a disintegration $\{\pi_\xi\}_{\xi \in \mathcal{F}}$ of π with respect to $\mathcal{O}_{mi}$.

4.1. THEOREM. *ρ_{mi} is π_ξ-almost surely constant on π-almost all fibers ξ of $\mathcal{O}_{mi}$.*

PROOF. For each n, let $k_n = \sup\{k : \rho_{nk} \leqq \rho_{mi}\}$, and $\Delta_{n1} = \rho_{n1} - \rho_{n0} = \rho_{n1}, \cdots, \Delta_{n,k_n} = \rho_{n,k_n} - \rho_{n,k_n-1}$. It is a consequence of the ordering property that k_n is $\mathcal{O}_{mi}$-measurable. We shall show that for π-almost all fibers ξ, $\Delta_{n1}, \cdots, \Delta_{n,k_n}$ are independent relative to the measure π_ξ with $\sum_{i=1}^{k_n} \Delta_{ni} = \rho_{n,k_n}$ converging π_ξ-almost surely to ρ_{mi}. The desired result then follows from the central limit theorem.

Fix n and l. Let $\mathcal{P}$ be the sub-σ-field of $\mathcal{D}$ generated by sets of the form $\{\rho_{nl} < \rho \leqq \rho_{mi}, f(\rho) \in E\}$ as ρ ranges over $\mathcal{T}$ and E over Σ.

(a) $\mathcal{O}_{mi} = \mathcal{O}_{nl} \vee \mathcal{P}$

This follows from the decomposition

$$\{\rho \leqq \rho_{mi}, f(\rho) \in E\} = \{\rho \leqq \rho_{nl}, f(\rho) \in E\} \cup \{\rho_{nl} < \rho \leqq \rho_{mi}, f(\rho) \in E\}.$$

(b) $\mathcal{P} \subset \mathcal{O}_{nl} \vee \mathcal{D}^+_{\rho_{nl}}$

To prove (b), it suffices to show that, for each $\rho \in \mathcal{T}$, ρ is $\mathcal{O}_{nl} \vee \mathcal{D}^+_{\rho_{nl}}$-measurable on $\{\rho \geqq \rho_{nl}\}$: that is, $\{\rho \geqq \rho_{nl}, \rho \leqq s\} \in \mathcal{O}_{nl} \vee \mathcal{D}^+_{\rho_{nl}}$. To see this, observe that for each $\rho \in \mathcal{T}$ there is an integer m, together with rationals $r_1, \cdots, r_m$ and stopping times $0 = \rho_0 \leqq \cdots \leqq \rho_m = \rho$ for which $\rho_{i+1} = T(\rho_i, r_{i+1})$, $i = 0, \cdots, m-1$. Let $i^* = \sup\{i : 0 \leqq i \leqq m, \rho_i \leqq \rho_{nl}\}$. Note that $\{\rho > \rho_{nl}\} = \{i^* \leqq m-1\}$. We claim that ρ_{i^*+1} is $\mathcal{O}_{nl} \vee \mathcal{D}^+_{\rho_{nl}}$-measurable. The ordering property implies that i^* is $\mathcal{O}_{nl}$-measurable. On $\{i^* \leqq m-1\}$, $d(f(t), f(\rho_{i^*})) < r_{i^*}$ for $t \in [\rho_{i^*}, \rho_{nk})$. But then

$$\begin{aligned}\rho_{i^*+1} &= \inf\{t : t \geqq \rho_{i^*}, d(f(t), f(\rho_{i^*})) \geqq r_{i^*+1}\}\\ &= \inf\{t : t \geqq \rho_{nk}, d(f(t), f(\rho_{i^*})) \geqq r_{i^*+1}\}.\end{aligned}$$

Since i^* is $\mathcal{O}_{nk}$-measurable, it now follows easily that ρ_{i^*+1} is $\mathcal{O}_{nk} \vee \mathcal{D}^+_{\rho_{nk}}$-measurable on $\{i^* \leqq m-1\} = \{\rho > \rho_{nl}\}$. The corresponding measurability of $\rho_{i^*+2}, \cdots$, and finally of $\rho_m = \rho$ now follow easily.

Since X is Markov, $\mathcal{D}_{\rho_{nl}} \perp \mathcal{D}^+_{\rho_{nl}}$ given $f(\rho_{nl})$ (Lemma 2.6). It is clear that $\mathcal{O}_{nl} \subset \mathcal{D}_{\rho_{nl}}$, so it follows from Lemma 2.5(a) that $\mathcal{D}_{\rho_{nl}} \perp \mathcal{D}^+_{\rho_{nl}}$ given $f(\rho_{nl}) \vee \mathcal{O}_{nl} = \mathcal{O}_{nl}$. Applying Lemma 2.5(b), we have $\mathcal{D}_{\rho_{nl}} \perp \mathcal{D}^+_{\rho_{nl}} \vee \mathcal{O}_{nl}$ given $\mathcal{O}_{nl}$. Since $\mathcal{P} \subset \mathcal{D}^+_{\rho_{nl}} \vee \mathcal{O}_{nl}$, Lemma 2.5(a) implies $\mathcal{D}_{\rho_{nl}} \perp \mathcal{D}^+_{\rho_{nl}} \vee \mathcal{O}_{nl}$ given $\mathcal{O}_{nl} \vee \mathcal{P}$. But $\mathcal{O}_{nl} \vee \mathcal{P} = \mathcal{O}_{mi}$, so we have $\mathcal{D}_{\rho_{nl}} \perp \mathcal{D}^+_{\rho_{nl}}$ given $\mathcal{O}_{mi}$. Now fix k. Let $C_k = \{k_n = k\}$. The ordering property implies that k_n is constant on fibers of $\mathcal{O}_{mi}$. The conditional independence we have just demonstrated implies that, if $l < k$, the random vectors $(\Delta_{n1}, \cdots, \Delta_{nl})$ and $(\Delta_{n,l+1}, \cdots, \Delta_{nk})$ are independent relative to the measure π_ξ for π-almost all fibers ξ in C_k. It follows that, for π-almost all fibers ξ of $\mathcal{O}_{mi}$, $\{\Delta_{n1}, \cdots, \Delta_{n,k_n}\}_{n=1}^\infty$ is a triangular array of random variables with the members of each row π_ξ-independent. It is an easy consequence of Proposition 2.11 that $\sum_{k=1}^{k_n} \Delta_{nk} = \rho_{n,k_n} \to \rho_{mi}$ on D_0. Since $k_n = \sup\{k : \rho_{nk} \leqq \rho_{mi}\}$, it follows from the density of the $\rho_{nk}(f)$'s in $[0, \infty)$, hence in $[0, \rho_{mi}(f))$, that $\sum_{k=1}^{k_n} \Delta_{nk} = \rho_{n,k_n} \to \rho_{mi}$ as $n \to \infty$. Let ζ be such a fiber: that is, $\Delta_{n_1}, \cdots, \Delta_{nk_n}$ are independent relative to the measure π_ζ for each n. If we could show that $\max_{1 \leq k \leq k_n} \Delta_{nk} \to 0$ in π_ζ-measure as $n \to \infty$, the nomal convergence criterion (stated on page 316 of [5]) would imply that ρ_{mi} is normally distributed relative to the measure π_ζ. Since ρ_{mi} is non-negative, this is only possible if ρ_{mi} is degenerate, that is, if it is π_ζ-a.s. equal to a constant. Thus to establish this degeneracy for π almost all ζ, it suffices to show that $\max_{1 \leq k \leq k_n} \Delta_{nk} \to 0$ pointwise on D_0. (For $\pi(D_0) = 1$ by assumption, so $\pi_\zeta(D_0) = 1$ for π-almost all fibers ζ.) Suppose not. Then there is an $f \in D_0$ and a subsequence of n's with corresponding j_n's, $1 \leqq j_n \leqq k_n$ such that $\Delta_{n,j_n}(f) = \rho_{n,j_n}(f) - \rho_{n,j_n-1}(f) \geqq \delta$ for some $\delta > 0$. For such an n, the oscillation of f on the intervals $[\rho_{n,j_n-1}(f), \rho_{n,j_n}(f))$ cannot exceed $2\varepsilon_n$. Since these intervals are all contained in $I = [0, \rho_{mi}(f)]$, it follows from Proposition 2.9 that there is a subinterval of I on which f is constant. But $f \in D_0$, so this is a contradiction. So $\max_{1 \leq k \leq k_n} \Delta_{nk}$ does indeed go to 0 pointwise on D_0 as $n \to \infty$, which completes the proof of the theorem.

The set $\{\rho_{lj} \leqq \rho_{mi}\}$ belongs to $\mathcal{O}_{mi}$ by virtue of the ordering property, hence is a union of fibers of $\mathcal{O}_{mi}$. With minor amendments (and more subscripts) the proof of the last theorem demonstrates that ρ_{lj} is π_ξ-almost surely constant on π-almost all fibers ξ of $\mathcal{O}_{mi}$ belonging to $\{\rho_{lj} \leqq \rho_{mi}\}$. We state this as a corollary.

4.2. COROLLARY. *ρ_{lj} is π_ξ-almost surely constant on π-almost all fibers ξ of $\mathcal{O}_{mi}$ contained in $\{\rho_{lj} \leqq \rho_{mi}\}$.*

Since $\mathcal{O}$ is a countably generated sub-σ-field of $\mathcal{D}$, there is a disintegration $\{\pi_\xi\}_{\xi\in\mathcal{F}}$ of π with respect to $\mathcal{O}$ ($\mathcal{F}$ denotes the class of all fibers of $\mathcal{O}$). This is the disintegration implicitly referred to in the following corollary.

4.3. Corollary. *For π-almost all fibers of ξ of $\mathcal{O}$ there is a $g \in \xi$ such that π_ξ is the point mass on g, and for which, given $\rho \in \mathcal{T}$, $\rho(f) = \rho(g)$ for π_ξ-almost all f in ξ.*

Proof. The argument used in the proof of Theorem 4.1 demonstrates that, for any $\rho \in \mathcal{T}$, ρ is π_ξ-almost surely constant on π-almost all fibers ξ of $\mathcal{O}_\rho$. Suppose ξ' is a fiber of $\mathcal{O}$ for which $\pi_{\xi'}$ is a probability measure and on which each $\rho \in \mathcal{T}$ is $\pi_{\xi'}$-essentially constant. Let t_ρ be the $\pi_{\xi'}$-almost constant value of ρ on ξ'. Since $\pi_{\xi'}$ is a non-zero measure, there is at least one $g \in \xi'$ with $\rho(g) = t_\rho$ for all $\rho \in \mathcal{T}$. Let h be any member of ξ' with $\rho(h) = t_\rho$ for all $\rho \in \mathcal{T}$. Recall our assumption that g and h belong to D_1. Since the t_ρ's are dense in $[0,\infty)$, $h = g$. Thus $\pi_{\xi'}$ concentrates all its mass on g. This completes the proof of the corollary.

We now consider the case in which $X = (\Omega, \mathcal{A}, \mathcal{M}_t, X_t, P)$ has a transition function $\{P_t(x, E)\}$ which induces a standard process on D. Recall that this means that, if the P_x's are the probability measures on path space constructed via $P_t(x, E)$ in the usual way, then $P_x(D) = 1$ for all $x \in S$ and $(D, \mathcal{D}, \mathcal{D}_{t^+}, \eta_t, \theta_t, P_x)$ is a standard Markov process in the sense of Blumenthal and Getoor. To avoid conflicting notation, we denote the measure on $\mathcal{D}$ corresponding to the initial measure μ by π_μ rather than P_μ (although we continue to use P_x rather than π_{δ_x}).

4.4. Theorem. *Suppose that X is a Markov process whose paths have no intervals of constancy, and which has a transition function $P_t(x, E)$ inducing a standard process on D. Then there is a $G \subset D$ such that, for each $f \in G$, there is a function $\gamma(f, \cdot)$ on $[0,\infty)$ onto $[0,\infty)$ which is strictly increasing and continuous, and which satisfies the following properties for any initial measure μ. (We denote π_μ by π, and $\gamma(\cdot, s)$ by γ_s.)*

(1) *G is a union of fibers of $\mathcal{O}$ with $G \in \mathcal{D}$ and $\pi(G) = 1$.*

(2) *For each s and t, $\{\gamma_t \leqq s\}$ belongs to the π-completion of $\mathcal{D}_{s^+}$.*

(3) *For each s and t,*

$$\gamma_{s+t} - \gamma_s = \gamma_t \circ \theta_{\gamma_s} \tag{4.1}$$

π-almost surely.

(4) *Let $\{\pi_\xi\}_{\xi\in\mathcal{F}}$ be a disintegration of π relative to $\mathcal{O}$. Then for π-almost all fibers ξ of $\mathcal{O}$, there is a $g \in \xi$ for which $\pi_\xi = \delta_g$ and $f \circ \gamma(f, \cdot) = g$ for each $f \in \xi$.*

PROOF. We apply Proposition 2.13, with the sub-σ-field $\mathscr{E}$ referred to in its statement equal to $\mathcal{O}$, to obtain a system $\{P_{x,\xi}(A)\}$ of set functions for which $\{P_{x,\xi}\}_{\xi\in\mathscr{F}}$ is, for fixed x, a disintegration of P_x with respect to $\mathcal{O}$ (here $\mathscr{F}$ is the class of fibers of $\mathcal{O}$), with the property that for each $\mathscr{A}\in\mathscr{D}$, $P_{x,\xi}(A)$ is jointly measurable in (x,ξ). Since $\rho_{1,0}=0$, $f(0)$ is an $\mathcal{O}$-measurable function of f. If ξ is a fiber of $\mathcal{O}$, the value of $f(0)$ common to all $f\in\xi$ is denoted by $\xi(0)$. Clearly $\xi(0)$ is an $\mathcal{O}$-measurable function of ξ. Let $\mathscr{D}_0$ be a countable subfield of $\mathscr{D}$ which generates $\mathscr{D}$. We say that a fiber ξ of $\mathcal{O}$ is a *good* fiber if there is a $g\in D$ such that $P_{\xi(0),\xi}(A)=\delta_g(A)$ for each $A\in\mathscr{D}$. We set G equal to the union of good fibers.

Let μ be an initial measure, and set $\pi=\pi_\mu$. Let $\{\rho_\xi\}_{\xi\in\mathscr{F}}$ be a disintegration of π relative to $\mathcal{O}$.

(a) For each $A\in\mathscr{D}$, $P_{\xi(0),\xi}(A)$ is $\mathcal{O}$-measurable in ξ and equal π-almost surely to $\pi_\xi(A)$.

PROOF OF (a). Suppose $A\in\mathscr{D}$, $P_{x,\xi}(A)$ is $\Sigma\times\mathcal{O}$-measurable in (x,ξ) by (1) of Theorem 2.13. Since $x=\xi(0)$ is $\mathcal{O}$-measurable in ξ, $P_{\xi(0),\xi}(A)$ is $\mathcal{O}$-measurable in ξ. Suppose $E\in\mathcal{O}$. Then

$$\begin{aligned}\int_C \pi_\xi(A)\pi(d\xi) &= \pi(A\cap C)\\ &= \int P_x(A\cap C)\mu(dx)\\ &= \int\left(\int_C P_{x,\xi}(A)P_x(d\xi)\right)\mu(dx)\\ &= \int\left(\int_C P_{\xi(0),\xi}(A)P_x(d\xi)\right)\mu(dx)\\ &= \int\left(\int I_C(\xi)P_{\xi(0),\xi}(A)P_x(d\xi)\right)\mu(dx)\\ &= \int I_C(\xi)P_{\xi(0),\xi}(A)\pi(d\xi)\\ &= \int_C P_{\xi(0),\xi}(A)\pi(d\xi).\end{aligned}$$

The various steps are justified not only by the definition of π, $\{\pi_\xi\}$, $\{P_{x,\xi}\}$ and their properties but by the fact that $P_x(\{\xi:\xi(0)=x\})=P_x(\{f:f(0)=x\})$. This property, called *normality* ([1], page 30), is part of the assumption that

$(D, \mathcal{D}, \mathcal{D}_{t^+}, \eta_t, \theta_t, P_x)$ is a standard process ([1], page 45). Since C is an arbitrary member of $\mathcal{O}$, the proof of (a) is complete.

Clearly ξ is a good fiber iff $P_{\xi(0),\xi}(D) = 1$ and $P^2_{\xi(0),\xi}(A) = P_{\xi(0),\xi}(A)$ for each $A \in \mathcal{D}_0$. We thus conclude from (a) that $G \in \mathcal{D}$.

Since $\mathcal{D}$ is countably generated, it follows from (a) that, for π-almost all ξ, $P_{\xi(0),\xi}(A) = \pi_\xi(A)$ for all $A \in \mathcal{D}$. Corollary 4.3 shows that the set of fibers ξ for which $\pi_\xi = \delta_g$ for some $g \in D$ has π-measure 1. Thus the set of ξ's with $P_{\xi(0),\xi}(A) = \delta_g$ for some g has π-measure 1, which proves (1). If ξ is such a fiber, and if $f \in \xi$, it follows from Theorem 3.5 and Proposition 2.10 that there is a unique change of variable taking f into g, and we define $\gamma(f, \cdot)$ to be that change of variable. Thus $\gamma(f, t)$ is defined for all $f \in G$ and $t \in [0, \infty)$, and (4) holds.

For each $f \in G$, let $\tilde{\rho}_{li}(f) = \rho_{li}(g)$ if $\pi_\xi = \delta_g$ for the fiber ξ containing $\mathcal{O}$. It follows from Corollaries 4.2, 4.3, and 2.12 that $\tilde{\rho}_{li}$ is π-almost surely $\mathcal{O}_{ni}$-measurable; in fact it is easy to deduce from Corollaries 4.2 and 4.3 that any $\mathcal{O}_{ni}$-measurable version of $E_\pi(\rho_{li} \mid \mathcal{O}_{li})$ is π-almost surely equal to $\tilde{\rho}_{li}$. Since $f \circ \gamma(f, \cdot) = g$, it follows from part (a) of Lemma 3.4 that $\gamma(f, \tilde{\rho}_{li}(f)) = \rho_{li}(f)$ for $f \in G$.

(b) For each s and t,

$$\{\gamma_s \leqq t\} = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcup_{(l,i)} \left\{ s \leqq \tilde{\rho}_{li} \leqq s + \frac{1}{m}, \ \rho_{li} \leqq t + \frac{1}{n} \right\} \tag{4.2}$$

(this equality assumed to hold modulo a π-null set).

Proof of (b). Suppose first that $f \in G$, and that $\gamma(f, s) \leqq t$. Fix n. Since $\gamma(f, \cdot)$ is strictly increasing and continuous, there is an m with $\gamma(f, s + 1/m) \leqq t + 1/n$. Since the ρ_{li}'s are dense in $[0, \infty)$ (by 3.2(d)), there is an (l, i) with $\tilde{\rho}_{li}(f) \in [s, s + 1/m)$. Then

$$\rho_{li}(f) = \gamma(\tilde{\rho}_{li}(f)) \leqq \gamma(f, s + 1/m) \leqq t + 1/m.$$

Since n is arbitrary, f belongs to the right hand side of (4.2).

Suppose f belongs to the right hand side of (4.2), and that $f \in G$. Fix n. There is an m and an (l, i) for which $s \leqq \tilde{\rho}_{li}(f) \leqq s + 1/m$, and $\rho_{li}(f) \leqq t + 1/n$. Then

$$\gamma(f, s) \leqq \gamma(f, \tilde{\rho}_{li}(f)) = \rho_{li}(f) \leqq t + 1/n.$$

Since this holds for all n, $\gamma(f, s) \leqq t$. Thus the intersections of the two sides of (4.2) with G are identical. Since $\pi(G) = 1$, (4.2) holds modulo a π-null set.

We now establish property (2). The set $\{s \leqq \tilde{\rho}_{li} \leqq s + 1/m, \ \rho_{li} \leqq t + 1/n\}$ is

π-almost surely equal to $\{s \leqq Z_{li} \leqq s + 1/m,\ \rho_{li} \leqq t + 1/n\}$ where Z_{li} is an $\mathcal{O}_{li}$-measurable version of $E_\pi(\rho_{li} \mid \mathcal{O}_{li})$. But $\{s \leqq Z_{li} \leqq s + 1/m\} \in \mathcal{O}_{li} \subset \mathcal{D}_{\rho_{li}}$, so

$$\{s \leqq Z_{li} \leqq s + 1/m\} \cap \{\rho_{li} \leqq t + 1/n\} \in \mathcal{D}_{t+1/n}.$$

Since $\{s \leqq Z_{li} \leqq s + 1/m,\ \rho_{li} \leqq t + 1/n\}$ is non-increasing in n for fixed (l, i) and m,

$$\bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcup_{(l,i)} \{s \leqq Z_{li} \leqq s + 1/m, \rho_{li} \leqq t + 1/n\} =$$

$$= \bigcap_{n=N}^{\infty} \bigcup_{m=1}^{\infty} \bigcup_{(l,i)} \{s \leqq Z_{li} \leqq s + 1/m, \rho_{li} \leqq t + 1/n\}$$

for any N, hence belongs to $\mathcal{D}_{t+1/N}$ for each N, hence to $\mathcal{D}_{t^+}$. It follows from (b) that $\{\gamma_s \leqq t\}$ belongs to the π-completion of $\mathcal{D}_{t^+}$.

We now prove (3). We shall establish that, for each value of s, both f and $\theta_{\gamma_s} f$ belong to good fibers, and that if $\pi_\zeta = \delta_g$, where ζ is the fiber to which f belongs, then $\pi_{\zeta_s} = \delta_{\theta_s g}$, where ζ_s is the fiber to which θ_{γ_s} belongs. It then will follow that $\gamma(\theta_{\gamma_s} f, \cdot)$ is the change of variable which takes $\theta_{\gamma_s} f$ into $\theta_s g$. This (unique) change of variable is easily seen to be the map $t \to \gamma(f, s + t) - \gamma(f, s)$. The proof that the translate of f by θ_{γ_s} corresponds to the translate of g by s is somewhat involved, but is basically an application of the strong Markov property.

Fix $s \in [0, \infty)$. Let ϕ denote θ_{γ_s}. Let μ_s be the distribution of η_{γ_s} that is, $\mu_s(E) = \pi(\{f : f(\gamma(f, s)) \in E\})$. Since γ_s is a $\mathcal{D}_{t^+}$-stopping time by virtue of (2), a fairly routine argument using the strong Markov property shows that $\pi \circ \phi^{-1} = \pi_{\mu_s}$ (see the proof of corollary 8.5 on page 39 of [1]).

(c) If Z is a bounded, $\mathcal{D}$-measurable function on D, then

$$E_\pi(Z \circ \phi \mid \mathcal{O}) = E_{\pi \circ \phi^{-1}}(Z \mid \mathcal{O}) \circ \phi \qquad \pi\text{-a.s.} \tag{4.3}$$

Proof of (c). For each $O \in \mathcal{O}$, $\phi^{-1}(O) = \{f : f \in G, \phi(f) \in \mathcal{O}\}$, and $\phi^{-1}(\mathcal{O}) = \{\phi^{-1}(0) : 0 \in \mathcal{O}\}$. Since G is a union of fibers of $\mathcal{O}$ with $\pi(G) = 1$, the identity

$$E_\pi(Z \circ \phi \mid \phi^{-1}(\mathcal{O})) = E_{\pi \circ \phi^{-1}}(Z \mid \mathcal{O}) \circ \phi, \tag{4.4}$$

well known if ϕ is $\mathcal{D}$-measurable and defined on all of G, holds for $\phi = \gamma_s$. Thus it suffices to prove

$$E_\pi(Z \circ \phi \mid \mathcal{O}) = E_\pi(Z \circ \phi \mid \phi^{-1}(\mathcal{O})). \tag{4.5}$$

Let $\mathcal{Q} = \mathcal{D}_{\gamma_s} \vee \phi^{-1}(\mathcal{O})$. By virtue of Lemma 3.6, $\mathcal{O} \subset \mathcal{D}_{\gamma_s} \vee \phi^{-1}(\mathcal{O})$. We now show that $E_\pi(Z \circ \phi \mid \phi^{-1}(\mathcal{O})) = E\ \ (Z \circ \phi \mid \mathcal{Q})$. Relative to π, $\mathcal{D}^+_{\gamma_s}$ is conditionally independent of $\mathcal{D}_{\gamma_s}$ given η_{γ_s}. But η_{γ_s} is π-a.s. $\phi^{-1}(\mathcal{O})$-measurable and $\phi^{-1}(\mathcal{O}) \subset$

$\mathcal{D}^+_{\gamma_s}$ (modulo π), so $\mathcal{D}^+_{\gamma_s}$ is conditionally independent of $\mathcal{D}^+_{\gamma_s}$ given $\eta_{\gamma_s} \vee \phi^{-1}(\mathcal{O}) = \phi^{-1}(\mathcal{O})$ by virtue of part (a) of Lemma 2.5. Since $Z \circ \phi$ is π-a.s. $\mathcal{D}^+_{\gamma_s}$-measurable, it follows that

$$E_\pi(Z \circ \phi \mid \mathcal{Q}) = E_\pi(Z \circ \phi \mid \mathcal{D}_{\gamma_s} \vee \phi^{-1}(\mathcal{O})) = E_\pi(Z \circ \phi \mid \phi^{-1}(\mathcal{O})).$$

We now show that $\phi^{-1}(\mathcal{O}) \subset \mathcal{O}$ (modulo π). Suppose that $\xi \subset G$. By (4) there is a $g \in \xi$ such that $f \circ \gamma(f, \cdot) = g$ for all $f \in \xi$. Fix $f \in \xi$, and denote $\gamma(f, \cdot)$ by σ. For each $t \in [0, \infty)$, let $f_s(t) = (\theta_{\gamma_s} f)(t) - f(\gamma_s)$, $g_s(t) = (\theta_s g)(t) - g(s)$, and $\sigma_s(t) = \gamma(f, t+s) - \gamma(s)$. In the proof of Lemma 3.4, we observed that $f_s \circ \sigma_s = g_s$. Since $f(\gamma(f, s)) = g(s)$, it follows that $\theta_{\gamma_s} f \circ \sigma_s = \theta_s g$. It follows from Theorem 3.5 that $\theta_{\gamma_s} f \equiv \theta_s g \pmod{\mathcal{O}}$; hence if f_1 and f_2 are both in ξ, $\theta_{\gamma_s} f_1 = \theta_{\gamma_s} f_2 \pmod{\mathcal{O}}$. This shows that for π-almost all fibers ζ of $\mathcal{O}$, $\phi^{-1}(\zeta)$ is a union of fibers of $\mathcal{O}$, which in turn shows that if $0 \in \mathcal{O}$, $\phi^{-1}(0)$ is, modulo a π-null set, a union of fibers of $\mathcal{O}$. Since $\phi^{-1}(\mathcal{O})$ is $\mathcal{D}$-measurable, it follows from Corollary 2.12 that $\phi^{-1}(0)$ is in the π-completion of $\mathcal{O}$.

Since $\mathcal{Q} \supset \mathcal{O} \supset \phi^{-1}(\mathcal{O})$, and since $E(Z \circ \phi \mid \mathcal{Q}) = E(Z \circ \phi \mid \sigma^{-1}(\mathcal{O}))$, it follows that $E(Z \circ \phi \mid \mathcal{O}) = E(Z \circ \phi \mid \phi^{-1}(\mathcal{O}))$, and (c) is proved.

The fact that two members of the same fiber $\xi \subset G$ are equivalent modulo $\mathcal{O}$ implies that the direct image $\phi(\xi)$ of ξ under ϕ is a subset of some fiber of $\mathcal{O}$. We denote this fiber by $\phi^*(\xi)$.

Let $\{\pi^s_\xi\}$ be a disintegration of $\pi \circ \phi^{-1}$ relative to $\mathcal{O}$. For each bounded, $\mathcal{D}$-measurable Z on D, we use the disintegrations $\{\pi^s_\xi\}$ and $\{\pi_\xi\}$ to produce $\mathcal{O}$-measurable versions of $E_\pi(Z \circ \phi \mid \mathcal{O})$ and $E_{\pi \circ \phi^{-1}}(Z \mid \mathcal{O})$ respectively. Then we conclude from (c) that the set of ξ's in G for which

$$\int Z \circ \phi d\pi_\xi = \int Z d\pi^s_{\phi^*(\xi)} \tag{4.6}$$

is of π-measure 1. Indeed, if $\mathcal{Z}$ is a countable collection of such Z's, the set of ξ's in G for which (4.6) holds for each $Z \in \mathcal{Z}$ is of π-measure 1.

Since $\pi \circ \phi^{-1} = \pi_{\mu_s}$, $\pi(\phi^{-1}(G)) = 1$ by (1), so $\pi(G \cap \phi^{-1}(G)) = 1$. Note that if $\xi \in G \cap \phi^{-1}(G)$, then $\xi \in G$ and $\phi^*(\xi) \subset G$. By (4), for π-almost all ξ, π_ξ is a point mass concentrating on ξ. It also follows from (4) that π^s_ξ is a point mass on ξ for $\pi \circ \phi^{-1}$-almost all ξ; that is, $\pi^s_{\phi^*(\xi)}$ is a point mass concentrating on $\phi^*(\xi)$ for π-almost all ξ. Thus the set H of fibers ξ for which $\xi \in G$, $\phi^*(\xi) \in G$, π_ξ is a point mass on ξ and $\pi_{\phi^*(\xi)}$ is such *point mass on* $\phi^*(\xi)$, and for which (4.6) holds for each $Z \in \mathcal{Z}$, has π-measure 1. Suppose $\xi \in H$. Then $\pi_\xi = \delta_g$ where (by virtue of (4)) for each $f \in \xi$, $\gamma(f, \cdot)$ is a change of variable taking f into g, and $\pi^s_{\phi^*(\xi)} = \delta_{g'}$, where, for each $f' \in \phi^*(\xi)$, $\gamma(f', \cdot)$ is a change of variable taking f'

into g'. By (4.6), we have $Z(\phi(g)) = Z(g')$ for all $Z \in \mathscr{Z}$; and $\mathscr{Z}$ can be chosen rich enough to ensure that this last implies that $g' = \phi(g)$. Now, if $f \in \xi$, $\phi(f) \in \phi^*(\xi)$, and so $\gamma(\phi(f), \cdot)$ is a change of variable which takes $\phi(f)$ into $\phi(g)$. But $\phi(f) = \theta_{\gamma_s} f$, and $\phi(g) = \theta_{\gamma_s} g = \theta_s g$ (this last because $\gamma(g, s) = s$). But we observed in the argument leading to (d) that the change of variable taking $\theta_{\gamma_s} f$ to $\theta_s g$ is σ^*, where $\sigma^*(t) = \sigma(t+s) - \sigma(t) = \gamma(f, t+s) - \gamma(f, t)$. (We emphasize once more that the phraseology "*the* change of variable" is justified by Proposition 2.10.) We have shown that for a fixed s, there is a union of fibers H with $\pi(H) = 1$ such that, if $f \in H$, $\gamma(\theta_{\gamma_s} f, t) = \gamma(f, t+s) - \gamma(f, t)$ for all t. This establishes property (3), completing the proof of the theorem.

§5. The equivalence under time changes of processes with the same hitting probabilities

Suppose that $X = (\Omega, \mathscr{A}, \mathscr{M}_t, X_t, P)$ and $\tilde{X} = (\tilde{\Omega}, \tilde{\mathscr{A}}, \tilde{\mathscr{M}}_t, \tilde{X}_t, \tilde{P})$ are quasi-left-continuous stochastic processes with right continuous σ-fields, whose paths have no intervals of constancy, and which have identical state-dependent hitting probabilities in the sense of [2], definition 2.6. We extend previous notational conventions from X to $\tilde{X}$. In particular the symbol $\tilde{X}$ will be used to denote not only the process $(\tilde{\Omega}, \tilde{\mathscr{A}}, \tilde{\mathscr{M}}_t, \tilde{X}_t, \tilde{P})$ but also the map of $\tilde{\Omega}$ into D given by setting $\tilde{X}(\tilde{\omega})$ equal to the function whose value at t is $\tilde{X}_t(\tilde{\omega})$. We use $\tilde{\pi}$ to denote the distribution of the process $\tilde{X} = (\tilde{\Omega}, \tilde{\mathscr{A}}, \tilde{\mathscr{M}}_t, \tilde{X}_t, \tilde{P})$; that is, $\tilde{\pi} = \tilde{P} \circ \tilde{X}^{-1}$.

5.1. Lemma. *If X and $\tilde{X}$ have the same state-dependent hitting probabilities and inital distributions, then the restrictions of π and $\tilde{\pi}$ to $\mathscr{O}$ are identical.*

Proof. In definition 3.13 of [2] there is defined a sub-σ-field of $\mathscr{D}$, denoted there by $\mathscr{O}$. We denote it by $\mathscr{O}^*$, as $\mathscr{O}$ is used for a different purpose here. It is shown in [2] that if f and g are cadlag functions without intervals of constancy and if f and g are in the same fiber of $\mathscr{O}^*$, then there is a change of variable σ such that $f \circ \sigma = g$ (lemma 5.2 of [2]). It is easy to see that the converse holds: if f and g are in D_0, and there is a change of variable σ such that $g = f \circ \sigma$, then f and g are in the same fiber of $\mathscr{O}^*$. It follows from Theorem 3.5 that $\mathscr{O}$ and $\mathscr{O}^*$, considered as σ fields over D_0, have the same fibers. Both are countably generated, so Blackwell's theorem implies that $\mathscr{O} = \mathscr{O}^*$. It is shown in [2], however, that the restrictions of π and π to $\mathscr{O}^*$ are identical (theorem 3.14).

5.2. Theorem. *Let X and $\tilde{X}$ be quasi-left-continuous processes, whose sample paths are right continuous, continuous from the left, and have no intervals of constancy. Suppose that they have the same state-dependent hitting probabilities,* *and the same initial distribution. Suppose also that $\tilde{X}$ is a Markov process with a transition function $P_t(x, E)$ which induces a standard process on D. Then there is a continuous time change $\{\tau_t, t \geqq 0\}$ for X such that $(\Omega, \mathcal{A}, \mathcal{H}_{t^+}, X_{\tau_t}, P)$ is a Markov process with transition function $P_t(x, E)$, where, for each t, $\mathcal{H}_t$ is the sub-σ-field of $\mathcal{A}$ generated by all X_{τ_s} with $0 \leqq s \leqq t$. If X is Markov, then $(\Omega, \mathcal{A}, \mathcal{M}_{\tau_t}, X_{\tau_t}, P)$ is a Markov process with transition function $\tilde{P}_t(x, E)$.*

PROOF. Assume the hypothesis. Then X satisfies the hypothesis of Theorem 4.4. Let μ be the initial distribution common to X and $\tilde{X}$. Then $\tilde{\pi} = \tilde{P} \circ \tilde{X}^{-1}$ is the measure on $(D, \mathcal{D})$ determined by μ and the transition function $\tilde{P}(x, E)$. As before, $\pi = P \circ X^{-1}$. Let $\tilde{\gamma}(f, t)$ be the function described in Theorem 4.4. (We use "$\tilde{G}$" instead of G, and replace π by $\tilde{\pi}$ and π_ξ by $\tilde{\pi}_\xi$ respectively.) We define $\{\tau_t, t \geqq 0\}$ on X by setting $\tau_t(\omega) = \tilde{\gamma}(X(\omega), t)$ for each ω for which $X(\omega) \in \tilde{G}$. Since $\tilde{G}$ is a union of fibers of $\mathcal{O}$ with $\tilde{\pi}$-measure 1, and since the restrictions of π and $\tilde{\pi}$ to $\mathcal{O}$ are identical by virtue of Lemma 5.1, G has π-measure 1, so the domain $X^{-1}(G)$ of τ_t has P-measure 1. Since

$$\{\tau_t \leqq s\} = \{\omega : \tilde{\gamma}(X(\omega), t) \leqq s\} X^{-1}(\tilde{\gamma}_t \leqq s) \in X^{-1}(\mathcal{D}_s^+) \subset \mathcal{M}_s,$$

and since continuity in t of τ_t is a consequence of the corresponding property for $\tilde{\gamma}_t$, $\{\tau_t, t \geqq 0\}$ is indeed a continuous time change for X.

We abuse notation by using $\tilde{\gamma}$ to denote the map from $\tilde{G}$ into D defined by $[\tilde{\gamma}(f)](t) = f(\tilde{\gamma}(f, t))$, $t \geqq 0$, $f \in \tilde{G}$. It is easy to see that the distribution of $\{X_{\tau_t}\}$ is equal to $\pi \circ \tilde{\gamma}^{-1}$. Let $\{\pi_\xi\}$ and $\{\tilde{\pi}_\xi\}$ be disintegrations relative to $\mathcal{O}$ of π and $\tilde{\pi}$ respectively. By Theorem 4.4, for $\tilde{\pi}$-almost all fibers ξ of $\mathcal{O}$, there is a $\tilde{g}$ such that $\tilde{\pi}_\xi = \delta_{\tilde{g}}$ and $\tilde{\gamma}(f) = \tilde{g}$ for each $f \in \xi$. Since $\pi = \tilde{\pi}$ on $\mathcal{O}$, the π-measure of this set of fibers is the same as its $\tilde{\pi}$-measure, namely 1. Let $A \in \mathcal{O}$. For π-almost all of these fibers, π_ξ is a probability measure concentrating all its mass on ξ: then if $\tilde{\pi}_\xi = \delta_{\tilde{g}}$ with $\tilde{\gamma}(f) = \tilde{g}$,

$$\pi_\xi(\{f : \tilde{\gamma}(f) \in A\}) = \pi_\xi(\{f : f \in \xi, \tilde{\gamma}(f) \in A\}) = \delta_{\tilde{g}}(A) = \tilde{\pi}_\xi(A).$$

Using once more the identity of π and $\tilde{\pi}$ on $\mathcal{O}$, it follows that $\pi(\tilde{\gamma}(f) \in A) = \tilde{\pi}(A)$. Thus $\pi \circ \tilde{\gamma}^{-1} = \tilde{\pi}$. In other words, $\{X_{\tau_t}, t \geqq 0\}$ and $\{\tilde{X}_t, t \geqq 0\}$ have the same finite-dimensional distributions.

Suppose $s \geqq 0$, that $0 \leqq s_1 < \cdots < s_n \leqq s$, that $E_i \in \Sigma$, $i = 1, \cdots, n$, and let $C = \{\omega : X_{\tau_{s_1}} \in E_1, \cdots, X_{\tau_{s_n}} \in E_n\}$. Let $t \geqq 0$, and $E \in \Sigma$. Then

$$\begin{aligned}P(\{X_{\tau_{s+t}} \in E\} \cap C) &= \pi(\{\eta_{s+t} \in E, \eta_{s_1} \in E_1, \cdots, \eta_{s_n} \in E_n\}) \\ &= \pi(\tilde{\gamma}^{-1}\{\eta_{s+t} \in E, \eta_{s_1} \in E_1, \cdots, \eta_{s_n} \in E_n\}) \\ &= \tilde{\pi}(\{\eta_{s+t} \in E, \eta_{s_1} \in E_1, \cdots, \eta_{s_n} \in E_n\}) \\ &= \int_{\{\eta_{s_1} \in E_1, \cdots, \eta_{s_n} \in E_n\}} \tilde{P}_t(\eta_s, E)d\tilde{\pi} \\ &= \int_{\{\eta_{\gamma_{s_1}} \in E_1, \cdots, \eta_{s_n} \in E_n\}} \tilde{P}_t(\eta_{\gamma_s}, E)d\pi \\ &= \int_C \tilde{P}_t(X_{\tau_s}, E)dP.\end{aligned}$$

It follows that

$$P(X_{\tau_{s+t}} \in E \mid \mathscr{H}_s) = \tilde{P}_t(X_{\tau_s}, E) \qquad P\text{-a.s.} \tag{5.2}$$

For any bounded Σ-measurable function f on S, let $(\tilde{P}_t f)(x) = \int f(y)\tilde{P}_t(x, dy)$, $x \in S$. Suppose f is a bounded continuous function on S. Then $(\tilde{P}_t f)(\eta_s)$ is $\tilde{\pi}$-almost surely continuous in s by Theorem 2.8 (applied directly to the path process induced on D by $\{\tilde{P}_t(x, E)\}$). This implies that $(\tilde{P}_t f)(\eta_{\gamma_s})$ is π-almost surely continuous in s, hence that $(\tilde{P}_t f)(X_{\tau_s})$ is P-almost surely continuous in s. A routine sort of argument, similar to but simpler than the proof of theorem 8.11 on page 41 of [1], now shows that $(\Omega, \mathscr{A}, \mathscr{H}_{t^+}, X_{\tau_t}, P)$ is a Markov process having $\{P_t(x, E)\}$ as transition function.

Suppose $X = (\Omega, \mathscr{A}, \mathscr{M}_t, X_t, P)$ is Markov. It follows from the definition of $\{\tau_t, t \geqq 0\}$ and (4.1) that $\tau_{t+s} - \tau_s \in X^{-1}(\mathscr{D}^+_{\gamma_s}) \subset \mathscr{F}^+_{\tau_s}$. It follows from Corollary 2.4 that

$$P(X_{\tau_{s+t}} \in E \mid \mathscr{M}_{\tau_s}) = P(X_{\tau_{s+t}} \in E \mid X_{\tau_s}) \qquad P\text{-a.s.} \tag{5.3}$$

Since X_{τ_s} is $\mathscr{H}_s$-measurable, it follows from (5.2) that

$$P(X_{\tau_{s+t}} \in E \mid X_{\tau_s}) = \tilde{P}_t(X_{\tau_s}, E) \qquad P\text{-a.s.} \tag{5.4}$$

Thus we have, for each s, $t \in [0, \infty)$ and $E \in \Sigma$,

$$P(X_{\tau_{s+t}} \in E \mid \mathscr{M}_{\tau_s}) = P_t(X_{\tau_s}, E). \tag{5.6}$$

An argument identical to the one given above, using Theorem 2.8, enables us to conclude that $(\Omega, \mathscr{M}, \mathscr{M}_{\tau_t}, X_{\tau_t}, P)$ is a Markov process having $\tilde{P}_t(x, E)$ as a transition function. This completes the proof of the theorem.

5.3. COROLLARY. *Suppose*

$$X = (\Omega, \mathcal{M}, \mathcal{M}_t, X_t, \theta_t, P^x) \quad and \quad \tilde{X} = (\tilde{\Omega}, \tilde{\mathcal{M}}, \tilde{\mathcal{M}}_t, \tilde{X}_t, \tilde{\theta}_t, \tilde{P}^x)$$

are Markov processes, with state space (S, Σ), *in the sense of Blumenthal and Getoor. Suppose that both* X *and* $\tilde{X}$ *are standard, that the sample paths of both have no intervals of constancy, and that they have the same hitting distributions* (see page 234 of [1]). *Then there is a time change* $\{\tau_t, t \geqq 0\}$ *for* X *such that* $X = (\Omega, \mathcal{M}, \mathcal{M}_{\tau_t}, X_{\tau_t}, \theta_{\tau_t}, P^x)$ *is a strong Markov process with the same transition function as* $\tilde{X}$. *Furthermore,* $\{\tau_t, t \geqq 0\}$ *is the inverse of a continuous additive functional on* X.

PROOF. Since $\tilde{X}$ is standard, $P^\mu(D) = 1$ for any initial distribution μ. The transition function of $\tilde{X}$ therefore induces a standard process on D. Now apply Theorem 4.4 to obtain the function $\tilde{\gamma}(f, t)$ described there, and set $\tau_t(\omega) = \tilde{\gamma}(X(\omega), t)$, where $X(\omega)$ is the member of D described by the sample path $\{X_s(\omega), s \geqq 0\}$. Let μ be any initial measure. Let π_μ and $\tilde{\pi}_\mu$ be the measures induced on $(D, \mathcal{D})$ by μ and the transition functions of X and $\tilde{X}$ respectively. The set $\tilde{G}$ of f's for which $\tilde{\gamma}(f, \cdot)$ is defined has $\tilde{\pi}_\mu$ measure 1; since $\tilde{G}$ is a union of fibers of $\mathcal{O}$ it has π_μ-measure 1 by virtue of Lemma 5.1, which is applicable since the stochastic processes $(D, \mathcal{D}, \mathcal{D}_{t^+}, \eta_t, \pi_\mu)$ and $(D, \mathcal{D}, \mathcal{D}_{t^+}, \eta_t, \tilde{\pi}_\mu)$ have the same state-dependent hitting probabilities. Now, $\pi_\mu = P_\mu \circ X^{-1}$, where $P_\mu = \int P^x(\cdot)\mu(dx)$ (see page 25 of [1]). Therefore $X^{-1}(\tilde{G})$ has P_μ-measure 1. Since μ is arbitrary, $\tau_t(\omega)$ is defined (for all $t \geqq 0$) almost surely in ω; here we are using "almost surely" in the sense of definition 5.7 on page 27 of [1]. For each $x \in S$, let $\pi_x = \pi_\mu$ and $\tilde{\pi}_x = \tilde{\pi}_\mu$ with $\mu = \delta_x$. To show that $\tilde{X}$ is strong Markov with the same transition function as X amounts to showing that, for each $x \in S$, the stochastic process $(\Omega, \mathcal{M}, \mathcal{M}_{\tau_t}, X_{\tau_t}, P^x)$ is a Markov process (in the sense of Definition 2.2) with the same transition function as the stochastic process $(\tilde{\Omega}, \tilde{\mathcal{M}}, \tilde{\mathcal{M}}_t, \tilde{X}_t, \tilde{P}^x)$. But this follows from Theorem 5.2.

We complete the proof by showing that $\{\tau_t, t \geqq 0\}$ is the inverse of an additive functional for X. (Although the additive functional will be defined almost surely, the set of ω's on which it is defined may not include the entire domain of $\{\tau_t\}$.) We begin by applying Theorem 4.4 to X (not $\tilde{X}$ as in the first part of the proof) to obtain a function $\gamma(f, t)$ and a $G \subset D$ satisfying the properties stated in the theorem. Suppose $\xi \subset G \cap \tilde{G}$. Then ξ contains a g_ξ and a $\gamma(f, t)$ such that $f \circ \gamma(f, \cdot) = g_\xi$ and $f \circ \tilde{\gamma}(f, \cdot) = \tilde{g}_\xi$ for each $f \in \xi$. We set $\alpha(g_\xi, \cdot) = \gamma(\tilde{g}_\xi, \cdot)$.

Suppose μ is any initial measure. Let $\pi = \pi_\mu$. Since $\pi_\xi = \delta_{g_\xi}$ for π-almost all fibers ξ, $\pi(\{g_\xi : \xi \subset G \cap G\}) = 1$.

Since $g_\xi \circ \tilde{\gamma}(g_\xi, \cdot) = \tilde{g}_\xi$, and $\tilde{g}_\xi \circ \gamma(\tilde{g}_\xi, \cdot) = g_\xi$, $\alpha(g_\xi, \cdot) = \gamma(g_\xi, \cdot)^{-1}$, which shows that γ is the inverse of α on the domain $\{g_\xi : \xi \subset G \cap \tilde{G}\}$ of α.

We now show that for each s, t, and initial measure μ, $\{g_\xi : \xi \subset G \cap \tilde{G},$ $\alpha(g_\xi, t) \leqq s\}$ belongs to the completion of $\mathscr{D}_t^+$ under π_μ. Since

$$\alpha(g_\xi, t) \leqq s \Leftrightarrow \gamma(g_\xi, t)^{-1} \leqq s \Leftrightarrow t \leqq \gamma(g_\xi, s),$$

and since

$$\{f : t \leqq \gamma(f, s)\} = \{f : \gamma(f, s) < t\} \subset \mathscr{D}_t \subset \mathscr{D}_{t^+}, \bigcup\{g_\xi : \alpha(g_\xi, t) \leqq s\}$$

is the intersection of a member of $\mathscr{D}_{t^+}$ with the set $\{g_\xi : \xi \subset G \cap \tilde{G}\}$, which has π_μ-measure 1.

What remains is to show that, for each $s \geqq 0$ and initial measure μ,

$$\alpha_{s+t} = \alpha_s + \alpha_t \circ \theta_s \qquad \pi_\mu\text{-almost surely.} \tag{5.5}$$

Earlier we showed that if $\xi \in G$ then $\theta_{\gamma_s}\xi$ is a subset of a fiber ξ_s of $\mathcal{O}$, and for π_μ-almost all $\xi \subset G$, $\xi_s \subset G$ and $g_{\xi_s} = \theta_s g_\xi$. If we replace X by $\tilde{X}$ in the argument used to establish this, we see that for $\tilde{\pi}_\mu$-almost all $\xi \subset \tilde{G}$, hence for $\tilde{\pi}_\mu$-almost all $\xi \subset G \cap \tilde{G}$, $\xi_s \subset G$ and $\tilde{g}_{\xi_s} = \theta_s \tilde{g}_{\xi_s}$. Since $\pi_\mu = \tilde{\pi}_\mu$ on $\mathcal{O}$, it follows that for π_μ-almost all $\xi \subset G \cap \tilde{G}$, $\xi_s \subset G \cap \tilde{G}$, $g_{\xi_s} = \theta_s g_\xi$, and $\tilde{g}_{\xi_s} = \theta_s \tilde{g}_\xi$. For such an ξ,

$$\alpha(g_f, s+t) = \gamma(\tilde{g}_\xi, s+t) = \gamma(\tilde{g}_\xi, s) + \gamma(\theta_{\gamma_s}\tilde{g}_\xi, t) = \alpha(g_\xi, s) + \gamma(\theta_{\gamma_s}\tilde{g}_\xi, t).$$

But $\alpha(\theta_s g_s, t) = \gamma(\theta_s \tilde{g}_\xi, t) = \gamma(\tilde{g}_\xi, s+t) - \gamma(\tilde{g}_\xi, s)$. Thus $\alpha(\theta, g_\xi, t) = \gamma(\theta_{\gamma_s}\tilde{g}_\xi, t)$, and so $\alpha(g_\xi, s+t) = \alpha(g_\xi, s) + \alpha(\theta_s g_\xi, t)$. This proves (5.5).

If we now define $A(\omega, t) = \alpha(X(\omega), t)$ for all $\omega \in \Omega$ for which $X(\omega) \in G \cap \tilde{G}$, it follows from the properties just now demonstrated to hold for α that A is a continuous additive functional for X of which $\{\tau_t, t \geqq 0\}$ is almost surely the inverse. (For the definition of additive functional, see page 148 of [1], where the multiplicative functional M figuring in the definition given there is taken to be identically equal to 1.) This completes the proof of the theorem.

§6. Theorem 5.2 contains the theorem of Blumenthal, Getoor, and McKean ([1], page 234) on the equivalence under time change of two standard processes with identical hitting distributions — in the case in which both processes have identically infinite killing times and the sample paths of both have no intervals of constancy. As we mentioned earlier, we intend to consider the case of Markov processes with holding states ([1], page 91) in a paper now under preparation.

In [3], Dubins and Schwartz showed that a continuous fair process, whose sample paths have no intervals of constancy, can be time-changed into Brownian motion. It is not hard to see that a fair process is a stochastic process with state-dependent hitting probabilities identical to those of the Brownian motion process. Our Theorem 5.2 thus includes their result as well.

Added in proof. We have recently ([8]) sharpened not only the results proved in this paper but also the ones stated in the introduction. We refer the reader to the research announcement [9].

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